Extended $\mathcal{N} = 1$ super Yang-Mills theory

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ABSTRACT: We solve a generalization of ordinary $\mathcal{N} = 1$ super Yang-Mills theory with gauge group U(N) and an adjoint chiral multiplet X for which we turn on both an arbitrary tree-level superpotential term $\int \mathrm{d}^2\theta \, \mathrm{Tr}\, W(X)$ and an arbitrary field-dependent gauge kinetic term $\int \mathrm{d}^2\theta \, \mathrm{Tr}\, V(X)W^\alpha W_\alpha$. When $W = 0$, the model reduces to the extended Seiberg-Witten theory recently studied by Marshakov and Nekrasov. We use two different points of view: a "macroscopic" approach, using generalized anomaly equations, the Dijkgraaf-Vafa matrix model and the glueball superpotential; and the recently proposed "microscopic" approach, using Nekrasov's sum over colored partitions and the quantum microscopic superpotential. The two formalisms are based on completely different sets of variables and statistical ensembles. Yet it is shown that they yield precisely the same gauge theory correlators. This beautiful mathematical equivalence is a facet of the open/closed string duality. A full microscopic derivation of the non-perturbative $\mathcal{N} = 1$ gauge dynamics follows.

KEYWORDS: Supersymmetric gauge theory, Gauge-gravity correspondence, Duality in Gauge Field Theories, Nonperturbative Effects.

Contents

1. General presentation

In two recent papers [1, 2], the author and collaborators have proposed a general micro-scopic approach to the solution of $\mathcal{N} = 1$ super Yang-Mills theories. In [1], it was explained how to apply Nekrasov's instanton technology [3] to $\mathcal{N} = 1$, including in the strongly cou-pled vacua. The formalism is based on a microscopic quantum superpotential W_{mic} whose

saddle points are in one-to-one correspondence with the full set of quantum vacua of the theory. In [2], explicit calculations were made up to two instantons, and it was shown that the results agree to this order with the predictions made using a totally different formalism based on the Dijkgraaf-Vafa matrix model and glueball superpotential [4, 5]. The main purpose of the present work is to present a proof of the exact equivalence between the two formalisms. This yields a full microscopic derivation of the exact results in $\mathcal{N} = 1$ gauge theories, including a non-perturbative justification of the Dijkgraaf-Vafa matrix model, the generalized anomaly equations and the Dijkgraaf-Vafa glueball superpotential.

1.1 The model

We shall focus, as in [1, 2], on the $\mathcal{N} = 1$ theory with gauge group $\mathrm{U}(N)$ and one adjoint chiral superfield X. The basic chiral operators are given by [5][1]

$$u_k = \mathrm{Tr}\, X^k \,, \tag{1.1}$$

$$v_k = -\frac{1}{16\pi^2} \mathrm{Tr}\, W^\alpha W_\alpha X^k \,, \tag{1.2}$$

where W^α is the vector chiral superfield which contains the gauge field and whose lowest component is the gluino field. Our main goal is to compute the gauge theory expectation values of the above operators, that are conveniently encoded in the generating functions

$$R(z) = \sum_{k \geq 0} \frac{\langle u_k \rangle}{z^{k+1}} \,, \tag{1.3}$$

$$S(z) = \sum_{k \geq 0} \frac{\langle v_k \rangle}{z^{k+1}} \,. \tag{1.4}$$

The theory is usually studied with an arbitrary tree-level superpotential $\mathrm{Tr}\, W(X)$ for the field X, which amounts to introducing arbitrary couplings to the scalar operators (1.1). For our purposes, it is extremely natural to introduce arbitrary couplings to the generalized glueball operators (1.2) as well. The tree-level lagrangian that we consider is thus of the form

$$L = \frac{1}{4\pi} \mathrm{Im} \int \mathrm{d}^2\theta \, \mathrm{Tr}\big(\tau(X) W^\alpha W_\alpha\big) + 2N \, \mathrm{Re} \int \mathrm{d}^2\theta \, \mathrm{Tr}\, W(X) \,, \tag{1.5}$$

where $\tau(X)$ and $W(X)$ are arbitrary polynomials in X. Instead of the field-dependent coupling τ, it is convenient to work with the polynomial V defined by

$$NV(z) = 2i\pi\tau(z) \,. \tag{1.6}$$

The lagrangian can then be written as

$$L = 2N \, \mathrm{Re} \int \mathrm{d}^2\theta \, \mathscr{W} \tag{1.7}$$

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[1]One can also introduce the chiral operators $\mathrm{Tr}\, W^\alpha X^k$, but they have zero expectation values due to Lorentz invariance. Non-trivial expectation values could be obtained by turning on Lorentz violating couplings $t_k^\alpha \, \mathrm{Tr}\, W_\alpha X^k$ in the tree-level superpotential. The resulting model can be studied straightforwardly using our methods, but we shall not do it here for the sake of simplicity.

with

$$\mathscr{W} = -\frac{1}{16\pi^2} \operatorname{Tr} V(X) W^\alpha W_\alpha + \operatorname{Tr} W(X). \tag{1.8}$$

Using the parametrization

$$V(z) = \lambda_{-1} + \sum_{k=0}^{d_V} \frac{\lambda_k}{k+1} z^{k+1}, \tag{1.9}$$

$$W(z) = \sum_{k=0}^{d_W} \frac{g_k}{k+1} z^{k+1}, \tag{1.10}$$

where the degrees d_V and d_W can be arbitrary, the superpotential (1.8) takes the form

$$\mathscr{W} = \lambda_{-1} v_0 + \sum_{k \geq 0} \frac{\lambda_k}{k+1} v_{k+1} + \sum_{k \geq 0} \frac{g_k}{k+1} u_{k+1}. \tag{1.11}$$

It is also useful to introduce a polynomial

$$t(z) = \sum_{k \geq 1} \frac{t_k}{k+1} z^{k+1} \tag{1.12}$$

satisfying the relation

$$t''(z) = NV(z) = 2i\pi\tau(z). \tag{1.13}$$

We shall use $t(z)$ or $V(z)$ interchangeably according to convenience, with

$$t_1 = N\lambda_{-1}, \quad t_k = \frac{N\lambda_{k-2}}{k(k-1)} \quad \text{for } k \geq 2. \tag{1.14}$$

Let us note that the usual instanton factor is given by

$$q = e^{t_1} = e^{N\lambda_{-1}}. \tag{1.15}$$

The model (1.5) has useful $U(1)_A$ and $U(1)_R$ global symmetries. The charges of the super-space coordinates θ^α, of the various fields and couplings, and of any superpotential w one may wish to consider are given in the following table,

	θ	W^α	X	u_k	v_k	g_k	$\lambda_k, \ k \geq 0$	q	w	
$U(1)_A$	0	0	1	k	k	$-k-1$	$-k-1$	$2N$	0	
$U(1)_R$	1	1	0	0	2	2	0	0	2 .	$\quad(1.16)$

When $W = 0$, the theory (1.5) has $\mathcal{N} = 2$ supersymmetry, with tree-level prepotential

$$\mathcal{F}_{\mathcal{N}=2}(X) = t(X). \tag{1.17}$$

This "extended" Seiberg-Witten theory was studied recently from the microscopic point of view by Marshakov and Nekrasov in [6], and their results will be particularly useful for us. When W is turned on, the $\mathcal{N} = 2$ moduli space is lifted, and the theory has a discrete set of vacua. Classically, the vacua are labeled by the numbers N_i of eigenvalues of X that

sit at given critical points of W. In such vacua, the gauge group is broken from U(N) down to U(N_1) $\times \cdots \times$ U(N_{d_W}). The *rank* of the vacuum is defined to be the number of non-zero integers N_i. Since a mass gap is created in each non-abelian unbroken factor, it coincides with the rank of the low energy gauge group U(1)^r. Moreover, chiral symmetry breaking generates an N_i-fold degeneracy for each U(N_i) factor. This will be explicitly demonstrated later. The quantum vacua corresponding to the integers N_i are thus labeled as $|N_i, k_i\rangle$ with $0 \leq k_i \leq N_i - 1$. Moreover, when $d_V \geq 1$, we may find that new vacua appear at the quantum level. These vacua go to infinity in field space in the classical limit.

The solution of the model (1.5) can be found using two a priori completely different approaches. One approach is motivated by the closed string dual of the gauge theory and is natural from the long-distance, macroscopic point of view. It is based on the Dijkgraaf-Vafa matrix model and the use of the glueball superpotential [4], or equivalently on the geometric transition picture and the flux superpotential in the dual closed string background [7]. We call this approach the *macroscopic formalism*. It is very difficult to justify this formalism from first principles. The second approach amounts to computing directly the relevant gauge theory path integrals. It is based on Nekrasov's sums over colored partitions [3] and the microscopic quantum superpotential [1]. This is natural from the short-distance point of view and thus we call this approach the *microscopic formalism*. The microscopic formalism provides rigorous, first-principle derivations of the non-perturbative gauge theory dynamics.

The goal of the present paper is to prove the equivalence between the two formalisms. Since the microscopic and macroscopic set-ups are equivalent to the open and closed string descriptions respectively, the mathematical equivalence we are going to derive is a beautiful facet of the open/closed string duality, in a rare case where a complete understanding can be achieved.

Let us now present briefly the main ingredients of the two formalisms. Full details and justifications will be given in later sections.

Notation. In the following, when we have an indexed family of parameters, we use a non-indexed boldface letter to represent all the parameters at once. For example $\boldsymbol{g}$ denotes all the g_k, and $\boldsymbol{t}$ all the t_k.

1.2 The macroscopic formalism

In the macroscopic formalism, the basic, natural variables are the generalized glueball operators (1.2). Their expectation values for fixed gluino condensates s_i in the unbroken factors of the gauge group are given in terms of averages over the statistical ensemble of a random hermitian matrix M of size $n \times n$ as

$$v_{k,\,\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \varepsilon) = N\varepsilon \left\langle\!\left\langle \boldsymbol{s} \right| \mathrm{Tr}\, X^k \left| \boldsymbol{s} \right\rangle\!\right\rangle_\varepsilon = \frac{N\varepsilon}{Z_{\mathrm{mac}}} \int \mathrm{d}\mu_{\mathrm{mac}}^M \, \mathrm{Tr}\, M^k \,, \qquad (1.18)$$

$$Z_{\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \varepsilon) = \int \mathrm{d}\mu_{\mathrm{mac}}^M = \exp \frac{\mathcal{F}_{\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \varepsilon)}{\varepsilon^2} \,. \qquad (1.19)$$

The "macroscopic" measure is given in terms of the components of the matrix M and the tree-level superpotential W,

$$\mathrm{d}\mu_{\mathrm{mac}}^{M} = \prod_{I=1}^{n} \mathrm{d}M_{II} \prod_{1 \leq I < J \leq n} \mathrm{d}\mathrm{Re}\, M_{IJ}\, \mathrm{d}\mathrm{Im}\, M_{IJ} \exp\left(-\frac{1}{\varepsilon} \mathrm{Tr}\, W(M)\right). \qquad (1.20)$$

The parameter ε, which can be interpreted as the strength of some particular supergravity background [8], is related to the size of the matrix M,

$$\varepsilon = \frac{s}{n}, \qquad (1.21)$$

with

$$s = \sum_i s_i. \qquad (1.22)$$

The precise prescription to compute (1.18) is as follows. If one wishes to describe a rank r vacuum, then one must expand the matrix integral around the corresponding classical saddle point $|N_1, \ldots, N_r\rangle$, by putting $n_i = s_i/\varepsilon$ eigenvalues of the matrix M at the critical point of W corresponding to the integer N_i. One then considers the large n, or small ε, 't Hooft's genus expansion.

From (1.18), (1.19) and (1.20), we obtain immediately relations valid for any ε,

$$v_{k,\,\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \varepsilon) = -Nk\frac{\partial \mathcal{F}_{\mathrm{mac}}}{\partial g_{k-1}}, \quad k \geq 1. \qquad (1.23)$$

These relations are the macroscopic analogue of the Matone's relations [9], see (1.48) and below.

From (1.18), we can get the generating function

$$S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}, \varepsilon) = \sum_{k \geq 0} \frac{v_{k,\,\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \varepsilon)}{z^{k+1}}. \qquad (1.24)$$

Most relevant to us will be the planar $\varepsilon \to 0$ limit

$$v_{k,\,\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}) = \lim_{\varepsilon \to 0} v_{k,\,\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \varepsilon), \qquad (1.25)$$

$$S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}) = \lim_{\varepsilon \to 0} S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}, \varepsilon). \qquad (1.26)$$

Note the following important feature: the function S_{mac} does *not* depend on the parameters $\boldsymbol{t}$ that enter the tree-level lagrangian (1.5).

The next step is to introduce the macroscopic quantum superpotential W_{mac}, which is nothing but the glueball superpotential. In terms of the "macroscopic" one-form

$$\lambda_{\mathrm{mac}} = S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g})\, \mathrm{d}z, \qquad (1.27)$$

it is given by

$$W_{\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{t}) = \frac{1}{2i\pi} \oint_{\alpha} V\lambda_{\mathrm{mac}} - \sum_i N_i \frac{\partial \mathcal{F}_{\mathrm{mac}}}{\partial s_i}, \qquad (1.28)$$

where α will always denote a large contour at infinity in the z-plane. The expectation values of the operators (1.1) are given by

$$u_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g},\boldsymbol{t}) = k\frac{\partial W_{\mathrm{mac}}}{g_{k-1}}\,, \tag{1.29}$$

with associated generating function

$$R_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g},\boldsymbol{t}) = \sum_{k\geq 0}\frac{u_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g},\boldsymbol{t})}{z^{k+1}}\,. \tag{1.30}$$

Unlike S_{mac}, R_{mac} does depend, linearly, on the parameters $\boldsymbol{t}$.

The parameters $\boldsymbol{s}$ are determined by solving the equations

$$\frac{\partial W_{\mathrm{mac}}}{\partial s_i}\big(\boldsymbol{s} = \boldsymbol{s}^*\big) = 0\,. \tag{1.31}$$

These equations have in general several solutions, that are in one-to-one correspondence with the quantum vacua of fixed rank r. The on-shell generating functions

$$S^*_{\mathrm{mac}}(z;\boldsymbol{g},\boldsymbol{t}) = S_{\mathrm{mac}}(z;\boldsymbol{s}^*,\boldsymbol{g})\,, \tag{1.32}$$

$$R^*_{\mathrm{mac}}(z;\boldsymbol{g},\boldsymbol{t}) = R_{\mathrm{mac}}(z;\boldsymbol{s}^*,\boldsymbol{g},\boldsymbol{t})\,, \tag{1.33}$$

are conjectured to coincide with the corresponding gauge theory observables,

$$S(z;\boldsymbol{g},\boldsymbol{t}) = S^*_{\mathrm{mac}}(z;\boldsymbol{g},\boldsymbol{t})\,, \tag{1.34}$$

$$R(z;\boldsymbol{g},\boldsymbol{t}) = R^*_{\mathrm{mac}}(z;\boldsymbol{g},\boldsymbol{t})\,. \tag{1.35}$$

Of course both S^*_{mac} and R^*_{mac} depend non-linearly on $\boldsymbol{t}$ because $\boldsymbol{s}^*$ gets a non-trivial $\boldsymbol{t}$-dependence upon solving (1.31).

1.3 The microscopic formalism

In the microscopic formalism, the basic, natural variables are the operators (1.1). Their expectation values for fixed boundary conditions at infinity a_i for the field X,[2]

$$X_\infty = \mathrm{diag}(a_1,\ldots,a_N) = \mathrm{diag}\,\boldsymbol{a}\,, \tag{1.36}$$

are given in terms of averages over the statistical ensemble of random colored partitions endowed with a suitable generalized Plancherel measure as

$$u_{k,\,\mathrm{mic}}(\boldsymbol{a},\boldsymbol{t},\epsilon) = \big\langle\boldsymbol{a}\big|\,\mathrm{Tr}\,X^k\big|\boldsymbol{a}\big\rangle_\epsilon = \frac{1}{Z_{\mathrm{mic}}}\sum_{\vec{\mathsf{p}}}\mu^{\vec{\mathsf{p}}}_{\mathrm{mic}}\,u_{k,\vec{\mathsf{p}}}\,, \tag{1.37}$$

$$Z_{\mathrm{mic}}(\boldsymbol{a},\boldsymbol{t},\epsilon) = \sum_{\vec{\mathsf{p}}}\mu^{\vec{\mathsf{p}}}_{\mathrm{mic}} = \exp\frac{\mathcal{F}_{\mathrm{mic}}(\boldsymbol{a},\boldsymbol{t},\epsilon)}{\epsilon^2}\,. \tag{1.38}$$

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[2]We work in euclidean space-time.

A colored partition $\vec{\mathsf{p}}$ is a collection of N ordinary partitions p_i, $\vec{\mathsf{p}} = (\mathsf{p}_1, \ldots, \mathsf{p}_N)$, which are characterized by integers $p_{i,\alpha}$ satisfying

$$p_{i,1} \geq p_{i,2} \geq \cdots \geq p_{i,\tilde{p}_{i,1}} > p_{i,\tilde{p}_{i,1}+1} = 0 \,, \tag{1.39}$$

$$\sum_{\alpha=1}^{\tilde{p}_{i,1}} p_{i,\alpha} = |\mathsf{p}_i| \,. \tag{1.40}$$

The integer $|\mathsf{p}_i|$ is called the size of the partition p_i, and

$$|\vec{\mathsf{p}}| = \sum_{i=1}^{N} |\mathsf{p}_i| \tag{1.41}$$

is the size of the colored partition $\vec{\mathsf{p}}$. To each partition p_i, it is convenient to associate a Young tableau Y_{p_i} with $p_{i,\alpha}$ boxes in the row number α (the uppermost row being the last, shortest row). The number of boxes in the column number β is then denoted by $\tilde{p}_{i,\beta}$ (the rightmost column being the last, shortest column). The integers $\tilde{p}_{i,\beta}$ automatically satisfy

$$\tilde{p}_{i,1} \geq \tilde{p}_{i,2} \geq \cdots \geq \tilde{p}_{i,p_{i,1}} > \tilde{p}_{i,p_{i,1}+1} = 0 \,, \tag{1.42}$$

$$\sum_{\beta=1}^{p_{i,1}} \tilde{p}_{i,\beta} = |\mathsf{p}_i| \,. \tag{1.43}$$

In (1.37) and (1.38), the "microscopic" measure is given in terms of the integers characterizing the colored partition $\vec{\mathsf{p}}$ and the tree-level gauge kinetic term. Explicitly, we have

$$\mu_{\mathrm{mic}}^{\vec{\mathsf{p}}} = \left(\nu_{\mathrm{mic}}^{\vec{\mathsf{p}}} \right)^2 \tag{1.44}$$

with

$$\nu_{\mathrm{mic}}^{\vec{\mathsf{p}}} = \frac{1}{\epsilon^{|\vec{\mathsf{p}}|}} \prod_{i=1}^{N} \left[\prod_{\square_{(\alpha,\beta)} \in Y_{\mathsf{p}_i}} \frac{1}{p_{i,\alpha} - \beta + \tilde{p}_{i,\beta} - \alpha + 1} \prod_{j \neq i} \frac{1}{a_i - a_j + \epsilon(\beta - \alpha)} \right] \times$$

$$\prod_{i<j} \prod_{\alpha=1}^{\tilde{p}_{i,1}} \prod_{\beta=1}^{p_{j,1}} \frac{\left(a_i - a_j + \epsilon(\tilde{p}_{j,\beta} - \alpha - \beta + 1) \right) \left(a_i - a_j + \epsilon(p_{i,\alpha} - \beta - \alpha + 1) \right)}{\left(a_i - a_j + \epsilon(1 - \alpha - \beta) \right) \left(a_i - a_j + \epsilon(\tilde{p}_{j,\beta} - \alpha + p_{i,\alpha} - \beta + 1) \right)} \times$$

$$\exp\left(\frac{1}{2\epsilon^2} \sum_{k \geq 1} \frac{t_k}{k+1} u_{k+1,\vec{\mathsf{p}}} \right) \tag{1.45}$$

and

$$u_{k,\vec{\mathsf{p}}} = \sum_{i=1}^{N} \left[a_i^k + \sum_{\alpha=1}^{\tilde{p}_{i,1}} \left(\left(a_i + \epsilon(p_{i,\alpha} - \alpha + 1) \right)^k - \left(a_i + \epsilon(p_{i,\alpha} - \alpha) \right)^k \right. \right.$$

$$\left. \left. + \left(a_i - \epsilon\alpha \right)^k - \left(a_i - \epsilon(\alpha - 1) \right)^k \right) \right] \,. \tag{1.46}$$

The parameter ϵ can be interpreted as being the strength of some particular supergravity background, the so-called Ω-background [3]. It is different from the supergravity background governed by the parameter ε in the macroscopic formalism. In particular, ε is associated with a non-trivial space-time curvature whereas ϵ is not.

Let us note that

$$u_{2,\vec{\mathfrak{p}}} = \sum_{i=1}^{N} a_i^2 + 2\epsilon^2 |\vec{\mathfrak{p}}| \,. \tag{1.47}$$

Using (1.45), (1.44) and (1.15), this implies that the dependence in the instanton factor in the sums (1.37) and (1.38) is given by $q^{|\vec{\mathfrak{p}}|}$. Thus colored partitions of size k contribute to the kth instanton order.

From (1.37), (1.38) and (1.45), we obtain immediately a set of generalized Matone's relations [9], valid for any ϵ,

$$u_{k,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{t}, \epsilon) = 2k \frac{\partial \mathcal{F}_{\text{mic}}}{\partial t_{k-1}}, \quad k \geq 2\,. \tag{1.48}$$

The usual Matone's relation corresponds to $k = 2$, $t_{k'} = 0$ for $k' \geq 2$ and $\epsilon = 0$. It was shown to be valid at finite ϵ in [10].

From (1.37) we can get the generating function

$$R_{\text{mic}}(z; \boldsymbol{a}, \boldsymbol{t}, \epsilon) = \sum_{k \geq 0} \frac{u_{k,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{t}, \epsilon)}{z^{k+1}}\,. \tag{1.49}$$

Most relevant to us will be the limit $\epsilon \to 0$ of vanishing Ω-background

$$u_{k,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{t}) = \big\langle \boldsymbol{a} \big| \operatorname{Tr} X^k \big| \boldsymbol{a} \big\rangle = \lim_{\epsilon \to 0} u_{k,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{t}, \epsilon)\,, \tag{1.50}$$

$$R_{\text{mic}}(z; \boldsymbol{a}, \boldsymbol{t}) = \lim_{\epsilon \to 0} R_{\text{mic}}(z; \boldsymbol{a}, \boldsymbol{t}, \epsilon)\,. \tag{1.51}$$

Note the following important feature: the function R_{mic} does *not* depend on the parameters $\boldsymbol{g}$ that enter the tree-level lagrangian (1.5).

The next step is to introduce the microscopic quantum superpotential W_{mic} [1]. In terms of the "microscopic" one-form

$$\lambda_{\text{mic}} = z R_{\text{mic}}(z; \boldsymbol{a}, \boldsymbol{t}) \, \mathrm{d}z\,, \tag{1.52}$$

it is given by

$$W_{\text{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = \big\langle \boldsymbol{a} \big| \operatorname{Tr} W(X) \big| \boldsymbol{a} \big\rangle = \frac{1}{2i\pi} \oint_{\alpha} \frac{W \lambda_{\text{mic}}}{z}\,. \tag{1.53}$$

The expectation values of the operators (1.2) are given by

$$\begin{aligned}
v_{0,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) &= \frac{\partial W_{\text{mic}}}{\partial \lambda_{-1}}\,, \\
v_{k,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) &= k \frac{\partial W_{\text{mic}}}{\partial \lambda_{k-1}} \quad \text{for } k \geq 1\,,
\end{aligned} \tag{1.54}$$

or equivalently by

$$v_{k,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = \frac{N}{k+1} \frac{\partial W_{\text{mic}}}{\partial t_{k+1}}\,. \tag{1.55}$$

The associated generating function is

$$S_{\text{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = \sum_{k \geq 0} \frac{v_{k,\,\text{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t})}{z^{k+1}}\,. \tag{1.56}$$

Unlike R_{mic}, S_{mic} does depend, linearly, on the parameters $\boldsymbol{g}$.

The parameters $\boldsymbol{a}$ are determined by solving the equations

$$\frac{\partial W_{\mathrm{mic}}}{\partial a_i}(\boldsymbol{a} = \boldsymbol{a}^*) = 0 \,. \tag{1.57}$$

These equations have in general several solutions, that are in one-to-one correspondence with the full set of quantum vacua of the theory [1]. This is in sharp contrast with the equations (1.31), that yield the vacua for fixed values of the rank r only. The on-shell generating functions

$$R_{\mathrm{mic}}^*(z; \boldsymbol{g}, \boldsymbol{t}) = R_{\mathrm{mic}}(z; \boldsymbol{a}^*, \boldsymbol{t}) \,, \tag{1.58}$$

$$S_{\mathrm{mic}}^*(z; \boldsymbol{g}, \boldsymbol{t}) = S_{\mathrm{mic}}(z; \boldsymbol{a}^*, \boldsymbol{g}, \boldsymbol{t}) \,, \tag{1.59}$$

are equal to the the corresponding gauge theory observables,

$$R(z; \boldsymbol{g}, \boldsymbol{t}) = R_{\mathrm{mic}}^*(z; \boldsymbol{g}, \boldsymbol{t}) \,, \tag{1.60}$$

$$S(z; \boldsymbol{g}, \boldsymbol{t}) = S_{\mathrm{mic}}^*(z; \boldsymbol{g}, \boldsymbol{t}) \,. \tag{1.61}$$

Of course, both functions R_{mic}^* and S_{mic}^* have a complicated non-linear dependence on $\boldsymbol{g}$ that comes from solving (1.57).

1.4 Outline of the paper

The two formalisms described above have a very similar structure, with each statement in a given framework corresponding to another statement in the other framework. There is an obvious parallel between (1.18) and (1.37), (1.19) and (1.38), (1.24) and (1.49), (1.28) and (1.53), (1.31) and (1.57). We have indicated in each row of table 1 quantities that play analogous rôles in the two formalisms. This mapping will be justified and made more precise in the following sections. Similar but much more detailed tables are given at the end of the paper.

The formal structural similarities between the formalisms should not hide the fact that the macroscopic and microscopic approaches are both technically and conceptually very different. Clearly the matrix model integrals at the basis of the macroscopic formalism and the sums over colored partitions at the basis of the microscopic formalism are totally different objects. A very important point is that the microscopic formalism is an approach from first principles. The equations (1.60) and (1.61) must be true by construction. This is unlike their conjectured macroscopic analogues (1.35) and (1.34).

Our aim in the following will be to prove the equivalence between the two formalisms, which can be summarized mathematically by the two fundamental equations

$$\begin{aligned} R_{\mathrm{mac}}^*(z; \boldsymbol{g}, \boldsymbol{t}) &= R_{\mathrm{mic}}^*(z; \boldsymbol{g}, \boldsymbol{t}) \\ S_{\mathrm{mac}}^*(z; \boldsymbol{g}, \boldsymbol{t}) &= S_{\mathrm{mic}}^*(z; \boldsymbol{g}, \boldsymbol{t}) \end{aligned} \tag{1.62}$$

which must be valid in all the vacua of the theory. These equations are remarkable mathematical identities that make the link between two seemingly unrelated starting points to

Macroscopic formalism	Microscopic formalism
Glueballs $\boldsymbol{s}$	Scalars $\boldsymbol{a}$
Hermitian matrix M	Colored partition $\vec{\mathsf{p}}$
Curved background ε	Ω-background ϵ
Superpotential couplings $\boldsymbol{g}$	Prepotential couplings $\boldsymbol{t}$
Matrix model partition function $\mathcal{F}_{\mathrm{mac}}$	Prepotential $\mathcal{F}_{\mathrm{mic}}$
Macroscopic superpotential $W_{\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g},\boldsymbol{t})$	Microscopic superpotential $W_{\mathrm{mic}}(\boldsymbol{a},\boldsymbol{g},\boldsymbol{t})$
$S_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g})\,\mathrm{d}z$	$zR_{\mathrm{mic}}(z;\boldsymbol{a},\boldsymbol{t})\,\mathrm{d}z$
$R_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g},\boldsymbol{t})\,\mathrm{d}z$	$S'_{\mathrm{mic}}(z;\boldsymbol{a},\boldsymbol{g},\boldsymbol{t})\,\mathrm{d}z$

Table 1: Formal mapping between the macroscopic and microscopic formalisms.

perform the calculations. On the physics side, the two completely different-looking albeit equivalent formulations correspond to the open string (the microscopic set-up) and the closed string (the macroscopic set-up) descriptions of the same gauge theory.

The paper is organized as follows. In section 2, we discuss in details the macroscopic formalism. We establish the equivalence between the matrix model formulas (1.18), (1.19) and the generalized Konishi anomaly equations for our extended $\mathcal{N}=1$ theory (1.5). We compute explicitly the functions $S_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g})$ and $R_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g},\boldsymbol{t})$ from (1.18) and (1.29). We then study the critical points of W_{mac}, solving (1.31) in full generality. The result yields explicit expressions for S^*_{mac} and R^*_{mac}. In section 3, we focus on the microscopic formalism. Using results from Marshakov and Nekrasov [6] and the strategy developed in [1, 2], we compute explicitly $R_{\mathrm{mic}}(z;\boldsymbol{a},\boldsymbol{t})$ and $S_{\mathrm{mic}}(z;\boldsymbol{a},\boldsymbol{g},\boldsymbol{t})$. It turns out that $S_{\mathrm{mic}}(z)$ is an infinitely multi-valued analytic function, whereas the other generating functions are always two-valued. We then solve the equations (1.57), and provide a full microscopic derivation of the anomaly equations which are at the heart of the macroscopic approach. This includes the derivation of conjectures made in [2] about the generators of the equations and their algebra. Equations (1.62) then follow. Section 4 contains our conclusions and future prospects. We have also included two appendices. In appendix A, we present the proof of a generalization of the Riemann bilinear relations that plays an important rôle in the main text. In appendix B, we illustrate the solution of the extended model (1.5) in the particular case of the rank one vacua.

2. The macroscopic formalism

2.1 The anomaly equations, $S_{\mathbf{mac}}$ and the matrix model

We start from the generalized Konishi anomaly equations for the model (1.5). We do not try to justify these equations beyond the usual perturbative arguments [5] for the moment,

since our point of view is to develop the macroscopic formalism in this section using the usual hypothesis, which will be eventually proven by comparing with the results of the microscopic approach in section 3.

We thus follow [5] and consider, in perturbation theory, the variations

$$\delta_{L_n} X = -\zeta X^{n+1} \,, \tag{2.1}$$

$$\delta_{J_n} X = \frac{\zeta}{16\pi^2} W^\alpha W_\alpha X^{n+1} \,, \tag{2.2}$$

where ζ is an infinitesimal parameter. These variations are generated by the operators

$$L_n = -X^{n+1} \frac{\delta}{\delta X} \,, \quad J_n = \frac{1}{16\pi^2} W^\alpha W_\alpha X^{n+1} \frac{\delta}{\delta X} \cdot \tag{2.3}$$

They act on the observables (1.1) and (1.2) as

$$L_n \cdot u_m = -m u_{n+m} \,, \quad J_n \cdot u_m = -m v_{n+m} \,, \quad L_n \cdot v_m = -m v_{n+m} \,, \quad J_n \cdot v_m = 0 \tag{2.4}$$

and satisfy the algebra

$$[L_n, L_m] = (n-m)L_{n+m} \,, \quad [L_n, J_m] = (n-m)J_{n+m} \,, \quad [J_n, J_m] = 0 \,. \tag{2.5}$$

The last equation in (2.4) is a consequence of the anticommuting nature of the chiral vector superfield W^α in the chiral ring at the perturbative level. We refer the reader to [5] for details.

Performing the changes of variables corresponding to the variations (2.1) and (2.2) in the gauge theory path integral yield the following two equations

$$-NW'(z)R_{\text{mac}}(z) - NV'(z)S_{\text{mac}}(z) + 2R_{\text{mac}}(z)S_{\text{mac}}(z) + N^2\Delta_R(z) = 0 \,, \tag{2.6}$$

$$-NW'(z)S_{\text{mac}}(z) + S_{\text{mac}}(z)^2 + N^2\Delta_S(z) = 0 \,. \tag{2.7}$$

The polynomial terms $N^2\Delta_R$ and $N^2\Delta_S$ are necessary to make the equations consistent with the asymptotics

$$S_{\text{mac}}(z) \underset{z\to\infty}{\sim} \frac{v_{0,\text{mac}}}{z} \,, \quad R_{\text{mac}}(z) \underset{z\to\infty}{\sim} \frac{N}{z} \tag{2.8}$$

that follow from the definitions (1.24) and (1.30). The first two terms in the left hand side of (2.6) come from the variation of the tree-level superpotential (1.8). The first term in (2.7) has the same origin. Note however that the polynomial V does not contribute to (2.7) because $J_n \cdot v_m = 0$. The terms $2R_{\text{mac}}S_{\text{mac}}$ and S_{mac}^2 are generated by one-loop anomalous jacobians in the path integral, in strict parallel with the usual one-loop Konishi anomaly [11].

In the perturbative framework where (2.6) and (2.7) are derived, the variables $u_{k,\text{mac}}$ and $v_{k,\text{mac}}$ must satisfy algebraic constraints that follow from their definitions in terms of a finite-size $N \times N$ matrix X. For example, there exists polynomials $\mathscr{P}_{\text{pert},p}$ such that

$$u_{N+p,\text{mac}} = \mathscr{P}_{\text{pert},p}(u_{1,\text{mac}}, \ldots, u_{N,\text{mac}}), \quad p \geq 1 \,. \tag{2.9}$$

Similarly, only $v_{0,\,\mathrm{mac}}, \ldots, v_{N-1,\,\mathrm{mac}}$ are independent. It is not too difficult to show that (2.6) and (2.7) are consistent with (2.9) only if R_{mac} and S_{mac} coincide with their classical values [12]. This is an unorthodox way to rederive the standard perturbative non-renormalization theorem for the chiral operators expectation values.

In order to carry on with the macroscopic approach, we shall use the

NON-PERTURBATIVE ANOMALY CONJECTURE [12, 2]: *The non-perturbative corrections to (2.6) and (2.7) are such that they can be absorbed in a non-perturbative redefinition of the variables that enter the equations.*

One of the most important contribution of our work is to give in section 3 the first direct proof of this conjecture. For the moment we consider it as the basic assumption of the macroscopic formalism. So we can use (2.6) and (2.7), but with relations

$$u_{N+p,\,\mathrm{mac}} = \mathscr{P}_p(u_{1,\,\mathrm{mac}}, \ldots, u_{N,\,\mathrm{mac}}; q, \lambda_0, \ldots, \lambda_{d_V}) \qquad (2.10)$$

that can be a priori arbitrary as long as they are consistent with the symmetries (1.16).

Let us note that the non-perturbative anomaly conjecture implies that the equations (2.4) and (2.5) must get very strong quantum corrections [2]. We refer the reader to [12, 2] and to section 3.4 for a more extensive discussion of these conceptually very important points.

This being said, we can use (2.7) to find S_{mac}. Since the equation does not depend on the polynomial V, we find that S_{mac} does not depend on $\boldsymbol{t}$, as was claimed in section 1.2. Thus the function S_{mac} is the same as in the usual $\mathcal{N} = 1$ gauge theory studied in [5], and in particular (1.18) holds (it is a direct consequence of the fact that (2.7) coincides with the loop equation of the matrix model). Explicitly, (2.7) implies that

$$S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}) = \frac{N}{2}\Big(W'(z) - \sqrt{W'(z)^2 - 4\Delta_S(z)}\Big). \qquad (2.11)$$

The minus sign in front of the square root in (2.11) is found by using the asymptotics (2.8). The function $S_{\mathrm{mac}}(z)$ is a two sheeted function with $r \leq d_W$ branch cuts. The integer $d_W - r$ is given by the number of double roots of the polynomial $W'^2 - 4\Delta_S$,

$$W'(z)^2 - 4\Delta_S(z) = N_{d_W-r}(z)^2 y_{\mathrm{mac},\,r}^2, \qquad (2.12)$$

where N_{d_W-r} is a polynomial of degree $d_W - r$. We see that $S_{\mathrm{mac}}(z)$ is a meromorphic function on a genus $r - 1$ hyperelliptic curve of the form

$$\mathcal{C}_{\mathrm{mac},\,r}: \ y_{\mathrm{mac},\,r}^2 = \prod_{i=1}^{r}(z - w_i^-)(z - w_i^+). \qquad (2.13)$$

This curve, with some contours used in the main text, is depicted in figure 1. The configurations corresponding to a given value of r correspond to the description of the rank r vacua of the gauge theory. This can be straightforwardly checked by studying the classical limit, which in this formalism corresponds to $s \to 0$. In particular, classically $w_i^- = w_i^+ = w_i$ satisfies $W'(w_i) = 0$.
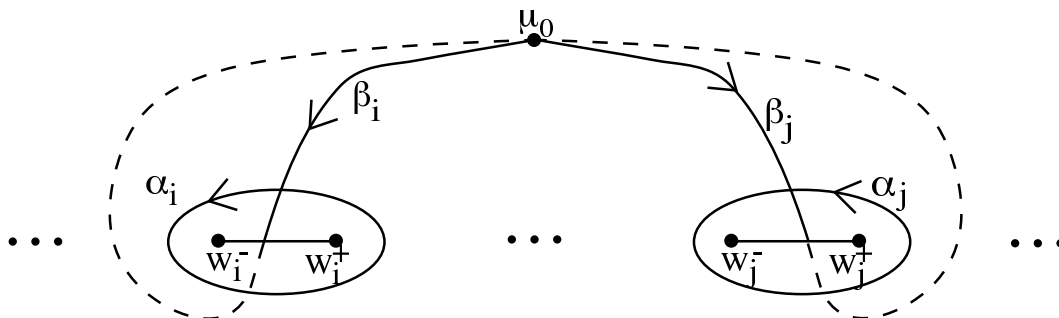
Figure 1: The curve $\mathcal{C}_{\mathrm{mac},r}$ defined by (2.13). On the first sheet (plain lines), the asymptotic conditions (2.8) are valid. We have depicted the contours α_i and β_i used in the main text. The contour at infinity is given by $\alpha = \sum_i \alpha_i$. The open contours β_i start and end at $z = \mu_0 \to \infty$, which corresponds to the points P_0 and Q_0 on the first and second sheets respectively.

The function S_{mac} given by (2.11) depends a priori on the d_W undetermined parameters that enter into Δ_S. The factorization condition (2.12) yields $d_W - r$ constraints, and thus there remains r free parameters. Using the relation between S_{mac} and the matrix model expectation values (1.18), these r free parameters can be related to the r parameters s_i defined after equation (1.22) by

$$s_i = \frac{1}{2i\pi N} \oint_{\alpha_i} \lambda_{\mathrm{mac}} \,, \tag{2.14}$$

where the meromorphic one-form λ_{mac} was defined in (1.27). Equations (2.11), (2.12) and (2.14) thus give the full prescription to compute the generating function $S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g})$. Note that the asymptotics (2.8) implies that

$$s = \sum_i s_i = \frac{1}{2i\pi N} \oint_{\alpha} \lambda_{\mathrm{mac}} = \frac{v_{0,\,\mathrm{mac}}}{N} \,, \tag{2.15}$$

or equivalently

$$v_{0,\,\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}) = v_{0,\,\mathrm{mac}}(\boldsymbol{s}) = N \sum_i s_i \,, \tag{2.16}$$

which is a particularly simple formula.

The fact that S_{mac} is single-valued on the curve (2.13) implies the following period integrals,

$$\oint_{\alpha_i} S'_{\mathrm{mac}} \, \mathrm{d}z = 0 \,, \tag{2.17}$$

$$\int_{\beta_i} S'_{\mathrm{mac}} \, \mathrm{d}z = N W'(\mu_0) \,. \tag{2.18}$$

These equations will have non-trivial counterparts in the microscopic formalism.

2.2 The glueball superpotential and $R_{\mathbf{mac}}$

The next important ingredient in the macroscopic formalism is the glueball superpotential given by (1.28).

2.2.1 Consistency with the anomaly equations and $R_{\mathbf{mac}}$

We are now going to check the $\boldsymbol{g}$- and $\boldsymbol{t}$-dependence of W_{mac} by showing that the equations

$$
\begin{aligned}
v_{0,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g}) &= \frac{\partial W_{\mathrm{mac}}}{\partial \lambda_{-1}}\,, \\
v_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g}) &= k\frac{\partial W_{\mathrm{mac}}}{\partial \lambda_{k-1}} \quad \text{for} \quad k \geq 1
\end{aligned}
\tag{2.19}
$$

and (1.29) are consistent with the anomaly equations (2.6) and (2.7). This is a generalization of the analysis made when $\lambda_k = 0$ for all $k \geq 0$ in [5].

The equations (2.19) are actually trivially satisfied. Indeed, all the dependence in $\boldsymbol{\lambda}$, or equivalently in $\boldsymbol{t}$, comes from the term containing V in (1.28). Using the fact that λ_{mac} does not depend on $\boldsymbol{t}$, we see that this term is precisely designed to be consistent with (2.19).

Checking the consistency of (1.29) is more interesting. Let us introduce the loop insertion operator

$$
\mathscr{L}_{\mathrm{mac}}(z) = -\sum_{k \geq 1} \frac{k}{z^{k+1}} \frac{\partial}{\partial g_{k-1}}\,.
\tag{2.20}
$$

Equations (1.23) and (1.29) are equivalent to

$$
S_{\mathrm{mac}}(z) = \frac{v_{0,\,\mathrm{mac}}}{z} + N\mathscr{L}_{\mathrm{mac}}(z) \cdot \mathcal{F}_{\mathrm{mac}}\,,
\tag{2.21}
$$

$$
R_{\mathrm{mac}}(z) = \frac{N}{z} - \mathscr{L}_{\mathrm{mac}}(z) \cdot W_{\mathrm{mac}}\,.
\tag{2.22}
$$

Applying the operator $\mathscr{L}_{\mathrm{mac}}(z)$ on (1.28) and using the above two equations together with (2.16) then yields

$$
R_{\mathrm{mac}}(z) = \frac{1}{N}\sum_{i=1}^{r} N_i \frac{\partial S_{\mathrm{mac}}(z)}{\partial s_i} - \frac{1}{2i\pi}\oint_{\alpha} \mathscr{L}_{\mathrm{mac}}(z)\cdot S_{\mathrm{mac}}(z')V(z')\,\mathrm{d}z'\,.
\tag{2.23}
$$

This formula can be written in a more elegant way. Let us introduce a canonical basis $\{h_i\}_{1 \leq i \leq r}$ of meromorphic one-forms on $\mathcal{C}_{\mathrm{mac},\,r}$, satisfying

$$
\frac{1}{2i\pi}\oint_{\alpha_j} h_i = \delta_{ij}\,,
\tag{2.24}
$$

that are holomorphic everywhere except at the points at infinity where they may have a simple pole. Explicitly, these forms can be written as

$$
h_i = \psi_i(z)\,\mathrm{d}z = \frac{p_i}{y_{\mathrm{mac},\,r}}\,\mathrm{d}z\,,
\tag{2.25}
$$

where the $p_i(z) = z^{r-1} + \cdots$ are monic polynomials of degree $r-1$ fixed by the conditions (2.24). Note that these conditions ensure that the $\{p_i\}_{1 \leq i \leq r}$ form a basis of the vector space of polynomials of degree at most $r-1$. By using the explicit expression (2.11) for S_{mac}, we get

$$
\frac{\partial S_{\mathrm{mac}}(z)}{\partial s_i} = \frac{N\partial\Delta_S(z)/\partial s_i}{\sqrt{W'(z)^2 - 4\Delta_S(z)}}\,.
\tag{2.26}
$$

Taking the derivative of the factorization condition (2.12), we see that the roots of N_{d_W-r} must also be roots of the polynomial $\partial\Delta_S(z)/\partial s_i$, and thus $\partial S_{\text{mac}}(z)/\partial s_i$ must be a linear combination of the functions ψ_i appearing in (2.25). Actually, taking the derivative of (2.14) with respect to s_j, and comparing with (2.24), we find that

$$\frac{\partial S_{\text{mac}}(z)}{\partial s_i} = N\psi_i(z)\,. \tag{2.27}$$

In terms of (1.27), this is equivalent to

$$\frac{\partial\lambda_{\text{mac}}}{\partial s_i} = Nh_i\,. \tag{2.28}$$

Equation (2.27) allows to write the first term in the right hand side of (2.23) in a suggestive way.

Let us now express $\mathscr{L}_{\text{mac}}(z) \cdot S_{\text{mac}}(z')$, by taking the derivative of (1.18) with respect to the couplings $\boldsymbol{g}$ using (1.20). We find

$$\mathscr{L}_{\text{mac}}(z) \cdot S_{\text{mac}}(z') = \frac{N}{\varepsilon^2}\bigg(\Big\langle\!\!\Big\langle\boldsymbol{s}\Big|\varepsilon\text{Tr}\,\frac{1}{z-X}\,\varepsilon\text{Tr}\,\frac{1}{z'-X}\Big|\boldsymbol{s}\Big\rangle\!\!\Big\rangle_\varepsilon$$
$$-\Big\langle\!\!\Big\langle\boldsymbol{s}\Big|\varepsilon\text{Tr}\,\frac{1}{z-X}\Big|\boldsymbol{s}\Big\rangle\!\!\Big\rangle_\varepsilon\Big\langle\!\!\Big\langle\boldsymbol{s}\Big|\varepsilon\text{Tr}\,\frac{1}{z'-X}\Big|\boldsymbol{s}\Big\rangle\!\!\Big\rangle_\varepsilon\bigg)\,. \tag{2.29}$$

The first term in (2.29) comes from the derivative of the numerator in (1.18), and the second term comes from the derivative of the partition function Z_{mac} in the denumerator. We have factored out explicitly $1/\varepsilon^2$ to emphasize the fact that in the planar $\varepsilon \to 0$ limit, it is the combination εTr that has a finite limit. In particular, the right hand side of (2.29) gets contributions from genus one, non-planar diagrams in the matrix model Feynman graph expansion. Plugging (2.29) and (2.27) in (2.23), we find the basic formula for R_{mac} in the macroscopic formalism,

$$R_{\text{mac}}(z;\boldsymbol{s},\boldsymbol{g},\boldsymbol{t}) = \sum_{i=1}^r N_i\psi_i(z) - \frac{N}{\varepsilon^2}\bigg(\Big\langle\!\!\Big\langle\boldsymbol{s}\Big|\varepsilon\text{Tr}\,\frac{1}{z-X}\,\varepsilon\text{Tr}\,V(X)\Big|\boldsymbol{s}\Big\rangle\!\!\Big\rangle_\varepsilon$$
$$-\Big\langle\!\!\Big\langle\boldsymbol{s}\Big|\varepsilon\text{Tr}\,\frac{1}{z-X}\Big|\boldsymbol{s}\Big\rangle\!\!\Big\rangle_\varepsilon\Big\langle\!\!\Big\langle\boldsymbol{s}\Big|\varepsilon\text{Tr}\,V(X)\Big|\boldsymbol{s}\Big\rangle\!\!\Big\rangle_\varepsilon\bigg)\,, \tag{2.30}$$

where the limit $\varepsilon \to 0$ is understood. Let us note that the constant term in V does not contribute to (2.30), and thus in the usual gauge theory with field-independent tree-level gauge coupling we have the much simpler formula $R_{\text{mac}} = \sum_i N_i\psi_i$. Turning on non-constant terms in V makes non-planar contributions to the matrix model relevant.

Equation (2.30) is conceptually interesting, but in order to compare with the anomaly equation (2.6) we need a more concrete formula. We can actually evaluate the integral in the right hand side of (2.23) more explicitly. We need to use the identity

$$\mathscr{L}_{\text{mac}}(z) \cdot S_{\text{mac}}(z') = \mathscr{L}_{\text{mac}}(z') \cdot S_{\text{mac}}(z)\,, \tag{2.31}$$

which follows either from (2.21) and $[\mathscr{L}_{\mathrm{mac}}(z), \mathscr{L}_{\mathrm{mac}}(z')] = 0$, or from the explicit expression (2.29). The expansions (2.20) and (1.9) then yield

$$\frac{1}{2i\pi} \oint_\alpha \mathscr{L}_{\mathrm{mac}}(z) \cdot S_{\mathrm{mac}}(z') V(z') \, \mathrm{d}z' = \frac{1}{2i\pi} \oint_\alpha \mathscr{L}_{\mathrm{mac}}(z') V(z') \, \mathrm{d}z' \cdot S_{\mathrm{mac}}(z)$$
$$= -\sum_{k \geq 0} \lambda_k \frac{\partial S_{\mathrm{mac}}(z)}{\partial g_k}, \qquad (2.32)$$

which implies that

$$R_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{t}) = \sum_{i=1}^r N_i \psi_i(z) + \sum_{k \geq 0} \lambda_k \frac{\partial S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g})}{\partial g_k}. \qquad (2.33)$$

The derivatives $\partial S_{\mathrm{mac}}(z)/\partial g_k$ can be computed from (2.11) in parallel with our previous computation of the derivatives $\partial S_{\mathrm{mac}}(z)/\partial s_i$ in (2.27) and above. Plugging the result in (2.33), and also using the explicit form of the ψ_i given in (2.25), we find that

$$R_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{t}) = \frac{N}{2}\left(V'(z) + \frac{D_R(z)}{y_{\mathrm{mac}, r}}\right), \qquad (2.34)$$

where $D_r(z)$ is a polynomial of degree $d_V + r = \deg V' + r$. Its coefficients are determined by the asymptotics (2.8), which yields $d_V + 2$ constraints, and by the conditions

$$\frac{1}{2i\pi} \oint_{\alpha_i} R_{\mathrm{mac}} \, \mathrm{d}z = N_i, \qquad (2.35)$$

which yield r additional constraints, only $r - 1$ of which are independent from the previous ones because $\sum_i N_i = N$. The period integrals (2.35) follow for example from (2.33) and (2.14).

We can now easily compare this result with the prediction from the anomaly equations. Using the solution to (2.7) in (2.6), we find

$$R_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{t}) = \frac{N}{2}\left(V'(z) + \frac{2\Delta_R(z) - W'(z)V'(z)}{\sqrt{W'(z)^2 - 4\Delta_S(z)}}\right). \qquad (2.36)$$

The anomaly equations do not put constraints on Δ_R, and thus R_{mac} in (2.36) has generically d_W branch cuts, each corresponding to a critical point of W. To describe vacua which classically correspond to having N_i eigenvalues of X sitting at the ith critical point, we need to impose the constraints (2.35), now for $1 \leq i \leq d_W$. When only r of the N_i are non-zero, the factorization (2.12) must take place, and the would-be poles of $R_{\mathrm{mac}}(z)$ at the roots of $N_{d_W - r}(z)$ must vanish (otherwise the corresponding N_j would be non-zero). The formula (2.36) then reduces to (2.34), as was to be shown.

Let us emphasize that (2.35) is non-trivial at the quantum level [12]. Indeed, there are many a priori consistent forms for the relations (2.10) that would violate the quantization conditions (2.35). The non-trivial statement is that the quantization conditions must be satisfied at the quantum level for the particular relations (2.10) for which the anomaly equations take the simple forms (2.6) and (2.7) (see the non-perturbative anomaly conjecture on page 12). In the way we have presented it, these constraints (2.35) look like additional inputs that are logically independent from the anomaly equations themselves. Remarkably, this is not the case [12, 13], see the discussion at the beginning of section 2.3.

2.2.2 Consistency with the U(1)$_\mathbf{R}$ symmetry

A useful consistency condition on W_{mac} comes from the U(1)$_\mathrm{R}$ symmetry in (1.16) [14],

$$
\begin{aligned}
W_{\mathrm{mac}}(\boldsymbol{s}, \boldsymbol{g}, \boldsymbol{t}) &= \sum_{k \geq 0} g_k \frac{\partial W_{\mathrm{mac}}}{\partial g_k} + \sum_{i=1}^{r} s_i \frac{\partial W_{\mathrm{mac}}}{\partial s_i} \\
&= \frac{1}{2i\pi} \oint_\alpha R_{\mathrm{mac}} W \, \mathrm{d}z + \sum_{i=1}^{r} s_i \frac{\partial W_{\mathrm{mac}}}{\partial s_i} \, .
\end{aligned}
\tag{2.37}
$$

Comparing with (1.28), we see that this a non-trivial formula in our formalism. We are going to present a first derivation based on matrix model identities along the lines of section 4.2.1 of [14]. Another derivation will be presented in section 2.3.3.

We start from the definition

$$
N\varepsilon \left\langle\!\!\left\langle \boldsymbol{s} \right| \operatorname{Tr} V(X) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle_\varepsilon = \frac{1}{2i\pi} \oint_\alpha V \lambda_{\mathrm{mac}} = \frac{N\varepsilon}{Z_{\mathrm{mac}}} \int \mathrm{d}\mu_{\mathrm{mac}}^M \operatorname{Tr} V(M) \,,
\tag{2.38}
$$

and we perform the infinitesimal variations

$$
\delta s_i = \zeta s_i \,, \quad \delta \varepsilon = \zeta \varepsilon \,.
\tag{2.39}
$$

Taking into account the variations of the global ε factor, of the numerator and of the denominator in the right hand side of (2.38), we obtain, in the $\varepsilon \to 0$ limit and by using (2.28),

$$
\begin{aligned}
\frac{1}{2i\pi} \oint_\alpha \sum_i s_i V \frac{\partial \lambda_{\mathrm{mac}}}{\partial s_i} &= \frac{N}{2i\pi} \oint_\alpha \sum_i s_i V h_i = N\varepsilon \left\langle\!\!\left\langle \boldsymbol{s} \right| \operatorname{Tr} V(X) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle_\varepsilon \\
&\quad + \frac{N}{\varepsilon^2} \Big(\left\langle\!\!\left\langle \boldsymbol{s} \right| \varepsilon \operatorname{Tr} W(X) \, \varepsilon \operatorname{Tr} V(X) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \boldsymbol{s} \right| \varepsilon \operatorname{Tr} W(X) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle \boldsymbol{s} \right| \varepsilon \operatorname{Tr} V(X) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle \Big).
\end{aligned}
\tag{2.40}
$$

The same reasoning starting from (1.19) shows that [14]

$$
\sum_i s_i \frac{\partial \mathcal{F}_{\mathrm{mac}}}{\partial s_i} - 2\mathcal{F}_{\mathrm{mac}} = \frac{1}{2i\pi N} \oint_\alpha W \lambda_{\mathrm{mac}} \,.
\tag{2.41}
$$

Taking the derivative with respect to s_i and using (2.27), we get

$$
-\frac{\partial \mathcal{F}_{\mathrm{mac}}}{\partial s_i} + \sum_j s_j \frac{\partial^2 \mathcal{F}_{\mathrm{mac}}}{\partial s_i \partial s_j} = \frac{1}{2i\pi} \oint_\alpha W h_i \,.
\tag{2.42}
$$

If we compute from the definition (1.28), we thus find

$$
\begin{aligned}
\sum_i s_i \frac{\partial W_{\mathrm{mac}}}{\partial s_i} &= \frac{N}{2i\pi} \oint_\alpha \sum_i s_i V h_i - \sum_{i,j} N_j s_i \frac{\partial^2 \mathcal{F}_{\mathrm{mac}}}{\partial s_i \partial s_j} \\
&= \frac{1}{2i\pi} \oint_\alpha V \lambda_{\mathrm{mac}} - \sum_i N_i \frac{\partial \mathcal{F}_{\mathrm{mac}}}{\partial s_i} - \frac{1}{2i\pi} \oint_\alpha W \sum_i N_i h_i \\
&\quad + \frac{N}{\varepsilon^2} \Big(\left\langle\!\!\left\langle \boldsymbol{s} \right| \varepsilon \operatorname{Tr} W(X) \, \varepsilon \operatorname{Tr} V(x) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \boldsymbol{s} \right| \varepsilon \operatorname{Tr} W(X) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle \boldsymbol{s} \right| \varepsilon \operatorname{Tr} V(X) \left| \boldsymbol{s} \right\rangle\!\!\right\rangle \Big) \\
&= W_{\mathrm{mac}} - \frac{1}{2i\pi} \oint_\alpha R_{\mathrm{mac}} W \, \mathrm{d}z \,.
\end{aligned}
\tag{2.43}
$$

To go from the first to the second equality in (2.43), we have used (2.40) and (2.42), and to go from the second to the third identity we have used (1.28) and (2.30).

2.3 The macroscopic quantum equations of motion

In the foregoing subsections, we have obtained explicit results for the generating functions $S_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g})$ and $R_{\mathrm{mac}}(z; \boldsymbol{s}, \boldsymbol{g}, \boldsymbol{t})$, see equations (2.8), (2.11), (2.12), (2.14), (2.34) and (2.35). We are now going to compute the on-shell generating functions $S_{\mathrm{mac}}^*(z; \boldsymbol{g}, \boldsymbol{t})$ and $R_{\mathrm{mac}}^*(z; \boldsymbol{g}, \boldsymbol{t})$, by solving (1.31) using the explicit formula (1.28).

2.3.1 On the consistency of the chiral ring

Before doing that, we would like to briefly discuss the following conceptually important question: is the formula (1.28) a new axiom of the macroscopic formalism, or is it enough to postulate the anomaly equations? We have seen in 2.2.1 that the $\boldsymbol{t}$- and $\boldsymbol{g}$-dependence of W_{mac} is fixed by comparing with the correlators deduced from the anomaly equations. There remains an undetermined piece of the form $w(\boldsymbol{s})$ in W_{mac}. Of course, this piece plays a crucial rôle in solving (1.31). Originally, it was thought that the precise form of this term, which encodes a crucial part of the non-perturbative gauge dynamics, needs to be postulated in addition to the anomaly equations themselves. Remarquably, it turns out that this is not necessary: the full $\boldsymbol{s}$-dependence of W_{mac} follows from consistency conditions once the anomaly equations have been postulated [14, 12, 13]. The same consistency conditions also imply that the quantization conditions (2.35) must be valid. This is a deep feature of the macroscopic formalism. We refer the reader to [13] for an extensive discussion, but let us briefly sketch the ideas involved.

For the vacua of rank $r = 1$, the argument is actually very simple [14]. In this case, $W_{\mathrm{mac}}(s, \boldsymbol{g}, \boldsymbol{t})$ depends on only one variable s, and the full s-dependence is then completely fixed by the consistency with the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry. Indeed, equation (2.37) is not invariant if we add to W_{mac} an arbitrary function of s. The condition (2.35) also follows immediately from the asymptotic behavior (2.8).

The case of the vacua of ranks $r > 1$ is much more interesting. One of the quantum equations of motion (1.31) is still related to the consistency with the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry, but there remains $r - 1$ independent constraints. In all the cases that have been studied (and we are going to show that this is true in the present extended model as well), they have the form of quantization conditions for the compact periods of $R_{\mathrm{mac}}\mathrm{d}z$,

$$\frac{1}{2i\pi} \oint_{\beta_i - \beta_j} R_{\mathrm{mac}} \, \mathrm{d}z \in \mathbb{Z} \,. \tag{2.44}$$

Combined with (2.35), we see that all the periods of $R_{\mathrm{mac}}\mathrm{d}z$ over the non-trivial cycles of the curve $\mathcal{C}_{\mathrm{mac},r}$ are integers. Equivalently, the quantum characteristic function

$$F_{\mathrm{mac}}^*(z; \boldsymbol{g}, \boldsymbol{t}) = \big\langle \det(z - X) \big\rangle \,, \tag{2.45}$$

which satisfies

$$\frac{F_{\mathrm{mac}}^{*\prime}(z; \boldsymbol{g}, \boldsymbol{t})}{F_{\mathrm{mac}}^*(z; \boldsymbol{g}, \boldsymbol{t})} = R_{\mathrm{mac}}^*(z; \boldsymbol{g}, \boldsymbol{t}) \,, \tag{2.46}$$

is well-defined (single-valued) on the curve $\mathcal{C}_{\mathrm{mac},r}$, together with R^*_{mac} and S^*_{mac}. The main general statement is as follows:

CHIRAL RING CONSISTENCY CONJECTURE [12]: *The anomaly equations are consistent with the existence of kinematical relations in the chiral ring of the form* (2.10) *if and only if the quantization conditions* (2.35) *and* (2.44) *are satisfied.*

Consistency is non-trivial because the relations (2.10) show that there is only a finite number of independent variables, and the anomaly equations yield an infinite set of constraints on this finite set of variables.

We have proven the above conjecture in the case of the ordinary $\mathcal{N}=1$ theory with a constant V and arbitrary W, including when $N_{\mathrm{f}} \leq 2N$ flavors of fundamental quarks with general couplings to X are added [13], and we believe that it is always true. It shows in particular that the full $\boldsymbol{s}$-dependence of W_{mac} is fixed by the algebraic consistency of the chiral ring.

2.3.2 The equations of motion

We need the following standard relation for the matrix model partition function (see for example [15] and references therein)

$$\frac{\partial \mathcal{F}_{\mathrm{mac}}}{\partial s_i} = \frac{1}{N} \int_{\beta_i} \lambda_{\mathrm{mac}} + 2s \ln \mu_0 - W(\mu_0)\,. \tag{2.47}$$

Using (1.28), (2.28) and (2.35), we thus obtain

$$\frac{\partial W_{\mathrm{mac}}}{\partial s_j} = \frac{N}{2i\pi} \oint_\alpha V h_j - \frac{1}{2i\pi} \sum_{i=1}^r \oint_{\alpha_i} R_{\mathrm{mac}}\, \mathrm{d}z \int_{\beta_i} h_j - 2N \ln \mu_0\,. \tag{2.48}$$

Let us introduce

$$H_j(P) = \int_{P_0}^P h_j + \ln \mu_0\,. \tag{2.49}$$

The analytic continuation $\hat{H}_j$ of H_j through the rth branch cut is given by[3]

$$H_j(z) + \hat{H}_j(z) = \int_{\beta_r} h_j + 2 \ln \mu_0\,, \tag{2.50}$$

which is found by integrating $\psi_j + \hat{\psi}_j = 0$ and finding the constant of integration by looking at $z \to \infty$. Let us now use the Riemann bilinear relation derived in appendix A with $F = R_{\mathrm{mac}}$ and $G = H_j$. The formula (A.12) applies because R_{mac} is a well-defined meromorphic function on $\mathcal{C}_{\mathrm{mac},r}$. Using

$$R_{\mathrm{mac}}(z) + \hat{R}_{\mathrm{mac}}(z) = NV'(z)\,, \tag{2.51}$$

which follows from (2.36), and also (2.24), we find

$$\int_{\beta_j} R_{\mathrm{mac}}\, \mathrm{d}z - \frac{1}{2i\pi} \sum_{i=1}^r \oint_{\alpha_i} R_{\mathrm{mac}}\, \mathrm{d}z \int_{\beta_i} h_j + \frac{1}{2i\pi} \oint_\alpha R_{\mathrm{mac}}\, \mathrm{d}z \int_{\beta_r} h_j =$$
$$-\frac{1}{2i\pi} \oint_\alpha \left[R_{\mathrm{mac}} H_j + \left(NV' - R_{\mathrm{mac}}\right)\left(\int_{\beta_r} h_j + 2 \ln \mu_0 - H_j\right)\right], \tag{2.52}$$

---

[3]See appendix A for details on the analytic continuation of functions like H_j.

or equivalently

$$\int_{\beta_j} R_{\mathrm{mac}}\, \mathrm{d}z = \frac{1}{2i\pi} \sum_{i=1}^{r} \oint_{\alpha_i} R_{\mathrm{mac}}\, \mathrm{d}z \int_{\beta_i} h_j + 2N \ln \mu_0 + \frac{1}{2i\pi} \oint_{\alpha} \left(NV'H_j - 2R_{\mathrm{mac}} H_j \right) \mathrm{d}z\,. \quad (2.53)$$

An integration by part immediately yields

$$\frac{1}{2i\pi} \oint_{\alpha} V'H_j\, \mathrm{d}z = -\frac{1}{2i\pi} \oint_{\alpha} V h_j + V(\mu_0)\,. \quad (2.54)$$

Using the asymptotics (2.8) and $H_j(z) \underset{z\to\infty}{\sim} \ln z$, we also get

$$\frac{1}{2i\pi} \oint_{\alpha} R_{\mathrm{mac}} H_j\, \mathrm{d}z = N \ln \mu_0\,. \quad (2.55)$$

Putting (2.48), (2.53), (2.54) and (2.55) together, we finally obtain

$$\frac{\partial W_{\mathrm{mac}}}{\partial s_i} = -\int_{\beta_i} R_{\mathrm{mac}}\, \mathrm{d}z + NV(\mu_0) - 2N \ln \mu_0\,. \quad (2.56)$$

The above formula is natural when one uses the parameters $\boldsymbol{t}$. However, from the physics point of view, a small refinement is needed, because t_1 is not really a good parameter. The good parameter is the instanton factor (1.15) $q = e^{t_1}$, and the theory must be invariant when $t_1 \to t_1 + 2i\pi$, which corresponds to a 2π shift of the ϑ angle. From (1.28), we know that t_1 enters W_{mac} only through a term $st_1 = s \ln q$. Thus we find that vacua related to each other by 2π shifts of the ϑ angle are described by different macroscopic superpotentials of the form $W_{\mathrm{mac}} + 2i\pi k s$, for $k \in \mathbb{Z}$. The most general vacua are thus obtained by looking at solutions of the equations

$$\int_{\beta_i} R_{\mathrm{mac}}^*\, \mathrm{d}z = NV(\mu_0) - 2N \ln \mu_0 + 2i\pi \mathbb{Z}\,. \quad (2.57)$$

As a last remark, let us note that (2.44) follows as a special case of (2.57).

2.3.3 The $\mathrm{U}(1)_{\mathbf{R}}$ symmetry revisited

It is instructive to rederive (2.37) starting from (2.56). Using (1.28), (2.14) and (2.47), (2.37) is equivalent to

$$\sum_{i=1}^{r} \left[\oint_{\alpha_i} R_{\mathrm{mac}}\, \mathrm{d}z \int_{\beta_i} S_{\mathrm{mac}}\, \mathrm{d}z - \oint_{\alpha_i} S_{\mathrm{mac}}\, \mathrm{d}z \int_{\beta_i} R_{\mathrm{mac}}\, \mathrm{d}z \right] =$$
$$N \oint_{\alpha} \left(S_{\mathrm{mac}} V - R_{\mathrm{mac}} W \right) \mathrm{d}z + 2i\pi N^2 \big(W(\mu_0) - sV(\mu_0) \big)\,. \quad (2.58)$$

We let the reader check that this is a consequence of (A.12), with $F = S_{\mathrm{mac}}$ and $G = \rho$ with

$$\rho(P) = \int_{P_0}^{P} R_{\mathrm{mac}}\, \mathrm{d}z + N \ln \mu_0\,. \quad (2.59)$$

Useful formulas to perform the check are

$$S_{\mathrm{mac}}(z) + \hat{S}_{\mathrm{mac}}(z) = NW'(z)\,, \quad (2.60)$$

which follows from (2.11), and

$$\rho(z) + \hat\rho(z) = NV(z) + \int_{\beta_r} R_{\text{mac}}\,\mathrm{d}z - NV(\mu_0) + 2N\ln\mu_0\,, \tag{2.61}$$

which follows by integrating (2.51). One also needs

$$\frac{1}{2i\pi}\oint_\alpha S_{\text{mac}}\rho\,\mathrm{d}z = N^2 s\ln\mu_0\,, \tag{2.62}$$

which is similar to (2.55).

2.4 The solution in the macroscopic formalism

Let us summarize our findings. The functions S^*_{mac} and R^*_{mac} are fully determined by the constraints

$$
\begin{aligned}
-NW'(z)R^*_{\text{mac}}(z) - NV'(z)S^*_{\text{mac}}(z) + 2R^*_{\text{mac}}(z)S^*_{\text{mac}}(z) + N^2\Delta_R(z) &= 0\,, \\
-NW'(z)S^*_{\text{mac}}(z) + S^*_{\text{mac}}(z)^2 + N^2\Delta_S(z) &= 0 \\
S^*_{\text{mac}}(z) \underset{z\to\infty}{\sim} \frac{Ns}{z}\,, \quad R^*_{\text{mac}}(z) \underset{z\to\infty}{\sim} \frac{N}{z}&\,, \\
\oint_{\alpha_i} R^*_{\text{mac}}\,\mathrm{d}z \in 2i\pi\mathbb{Z}\,, \quad \int_{\beta_i} R^*_{\text{mac}}\,\mathrm{d}z - NV(\mu_0) + 2N\ln\mu_0 &\in 2i\pi\mathbb{Z}\,.
\end{aligned}
\tag{2.63}
$$

The first three lines in (2.63) follow from the corresponding formulas (2.6), (2.7) and (2.8) for R_{mac} and S_{mac}, while the constraints in the fourth line follow from (2.35) and (2.57). The last equation is valid only on-shell, i.e. when $R_{\text{mac}} = R^*_{\text{mac}}$ and $S_{\text{mac}} = S^*_{\text{mac}}$. Note that the asymptotics (third line) and the quantization conditions (fourth line) in (2.63) provide enough constraints to fix the solution uniquely, up to a discrete ambiguity corresponding to the existence of a discrete set of vacua.

The quantization conditions in (2.63) are equivalent to the fact that the quantum characteristic function (2.45), (2.46) is single-valued on the same hyperellictic curve as S^*_{mac} and R^*_{mac}, with the asymptotics

$$F^*_{\text{mac}}(z) \underset{z\to\infty}{\sim} z^N\,, \quad \hat F^*_{\text{mac}}(z) \underset{z\to\infty}{\sim} \frac{e^{NV(z)}}{z^N}\,. \tag{2.64}$$

The function F has in general an essential singularity at Q_0. It is only in the case of the non-extended theory, for which V is a constant, that it is a meromorphic function together with S^*_{mac} and R^*_{mac}.

We refer the reader to appendix B, where the explicit solution for rank one vacua is discussed.

3. The microscopic formalism

In this section, we are going to solve the model using the microscopic formalism [1].

3.1 Marshakov-Nekrasov and R_{mic}

We start by computing the generating function R_{mic} which, in the microscopic formalism, is the simplest object. The main property that we are going to use is that R_{mic} does *not* depend on $\boldsymbol{g}$,

$$R_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = R_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{t}) \,. \tag{3.1}$$

This property, which is the analogue of the fact that S_{mac} does not depend on $\boldsymbol{t}$ in the macroscopic set-up, is manifest on the definition (1.37) and follows immediately from the localization formula [16, 2]. We can thus compute R_{mic} in the $\boldsymbol{g} = 0$ theory, which has $\mathcal{N} = 2$ supersymmetry. Fortunately for us, the extended $\mathcal{N} = 2$ theory, for arbitrary $\boldsymbol{t}$, has been studied recently by Marshakov and Nekrasov in [6], and so we can just borrow the result from them.[4]

The function $R_{\mathrm{mic}}(z)$ turns out to be a meromorphic function on the genus $N - 1$ hyperelliptic curve

$$\mathcal{C}_{\mathrm{mic}} : \ y^2 = \prod_{i=1}^{N} (x - x_i^-)(x - x_i^+) \,. \tag{3.2}$$

On $\mathcal{C}_{\mathrm{mic}}$, we define marked points P_0 and Q_0, as well as contours α_i, $\alpha = \sum_i \alpha_i$, β_i and $\delta_i = \beta_i - \beta_N$ is a way similar to what we have done on $\mathcal{C}_{\mathrm{mac}, r}$ in figure 1.[5] We have

$$R_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{t}) = \frac{N}{2}\left(V'(z) + \frac{E_R(z)}{y} \right) , \tag{3.3}$$

where E_R is a polynomial of degree $N + \deg V' = N + d_V$. The $N + d_V + 1$ coefficients in E_R and the $2N$ branching points of the curve (3.2) are determined by the following conditions [6]:

$$R_{\mathrm{mic}}(z) \underset{z \to \infty}{\sim} \frac{N}{z} \,, \tag{3.4}$$

$$\frac{1}{2i\pi} \oint_{\alpha_i} R_{\mathrm{mic}}\, \mathrm{d}z = 1 \,, \tag{3.5}$$

$$\int_{\beta_i} R_{\mathrm{mic}}\, \mathrm{d}z = N V(\mu_0) - 2N \ln \mu_0 + 2i\pi \mathbb{Z} \,, \tag{3.6}$$

$$a_i = \frac{1}{2i\pi} \oint_{\alpha_i} \lambda_{\mathrm{mic}} \,. \tag{3.7}$$

To write these equations, we have assumed that the curve (3.2) is not degenerate. This is always the case for large enough $|a_i - a_j|$, and the solution is then uniquely specified for any values of the a_i by analytic continuation in the $\boldsymbol{a}$-space. The one-form λ_{mic} is defined in (1.52). The first constraint (3.4) yields $d_V + 2$ conditions, the second constraint (3.5) yields $N - 1$ conditions that are independent from the previous ones, and the third and

---

[4]There is an unfortunate typo in Marshakov and Nekrasov that pollutes many of their formula. The typo first appears in their equation (3.2), where $\boldsymbol{t}'$ should be replaced by $\frac{1}{2}\boldsymbol{t}'$. We have corrected all their subsequent equations accordingly.

[5]We use the same names for the contours on different curves. Which curve we are referring to is always clear from the context.

fourth constraints (3.6) and (3.7) yield $N + N$ new conditions, for a total of $3N + d_V + 1$ independent conditions as needed. Let us note that the contours β_i are really defined modulo an integral linear combination of the α_j, which explains the term $+2i\pi\mathbb{Z}$ in (3.6).

The difference between the macroscopic and microscopic formalisms is here manifest. Equation (3.6) is valid *off-shell*, i.e. for any values of the parameters $\boldsymbol{a}$. The analogous equation (2.57) for R_{mac} is valid only *on-shell*, i.e. for the particular values $\boldsymbol{s}^*$ of the parameters $\boldsymbol{s}$ that make W_{mac} extremal. This is a very general feature: "simple" equations in a given formalism, valid off-shell, correspond to "complicated" equations in the other formalism, valid only on-shell. This is the mechanism that will allow to identify R_{mac}^* and R_{mic}^* on the one hand and S_{mac}^* and S_{mic}^* on the other hand, in spite of the fact that the off-shell functions have very different properties.

3.2 The generating function S_{mic}

Let us now compute the generating function for the generalized glueball operators S_{mic}. This function has been studied in details in [2]. In particular, a general formula for the $v_{k,\text{mic}}$ was derived (the derivation was made in the usual theory with $t_k = 0$ for $k \geq 2$, but it actually applies without change to the case of the extended theory),

$$v_{k,\text{mic}}(\boldsymbol{a},\boldsymbol{g},\boldsymbol{t},\epsilon) = \frac{N}{(k+1)(k+2)}\frac{1}{\epsilon^2}\Big(\big\langle\boldsymbol{a}\big|\operatorname{Tr} W(X)\operatorname{Tr} X^{k+2}\big|\boldsymbol{a}\big\rangle_\epsilon$$
$$- \big\langle\boldsymbol{a}\big|\operatorname{Tr} W(X)\big|\boldsymbol{a}\big\rangle_\epsilon\big\langle\boldsymbol{a}\big|\operatorname{Tr} X^{k+2}\big|\boldsymbol{a}\big\rangle_\epsilon\Big). \quad (3.8)$$

This can be conveniently rewritten in terms of S_{mic} as

$$S''_{\text{mic}}(z;\boldsymbol{a},\boldsymbol{g},\boldsymbol{t}) = \frac{N}{\epsilon^2}\Big(\big\langle\boldsymbol{a}\big|\operatorname{Tr}\frac{1}{z-X}\operatorname{Tr} W(X)\big|\boldsymbol{a}\big\rangle_\epsilon$$
$$- \big\langle\boldsymbol{a}\big|\operatorname{Tr}\frac{1}{z-X}\big|\boldsymbol{a}\big\rangle_\epsilon\big\langle\boldsymbol{a}\big|\operatorname{Tr} W(X)\big|\boldsymbol{a}\big\rangle_\epsilon\Big), \quad (3.9)$$

where the $'$ always means the derivative with respect to z. This formula, which is the microscopic analogue of (2.30), will be rederived shortly.[6] It shows that the subleading terms in the small ϵ expansion contribute to the glueball operators. Another contribution of [2] was to compute explicitly S_{mic} up to two instantons using (3.9) and the definitions given in 1.3. We are going to derive in the present section the *exact* formula for S_{mic}, by imitating the macroscopic derivation of R_{mac} in 2.2.1.

Let us start by explaining the origin of (3.9) in the present set-up. We shall need a few simple identities. We note that (1.46) implies that $u_{1,\vec{\mathsf{p}}}$ does not depend on the colored partition $\vec{\mathsf{p}}$,

$$u_{1,\vec{\mathsf{p}}} = \sum_{i=1}^{N} a_i. \quad (3.10)$$

---

[6]Note that (2.30) and (3.9) are actually valid at finite ε and ϵ respectively, even though we are only interested in the $\varepsilon \to 0$ and $\epsilon \to 0$ limits in the present paper.

From this we deduce that

$$u_{1,\,\mathrm{mic}}(\boldsymbol{a}, \boldsymbol{t}) = u_{1,\,\mathrm{mic}}(\boldsymbol{a}) = \sum_{i=1}^{N} a_i = a \,, \tag{3.11}$$

which is a particularly simple formula, as well as

$$\big\langle \boldsymbol{a} \big| \operatorname{Tr} X \operatorname{Tr} X^k \big| \boldsymbol{a} \big\rangle_\epsilon = \big\langle \boldsymbol{a} \big| \operatorname{Tr} X \big| \boldsymbol{a} \big\rangle_\epsilon \big\langle \boldsymbol{a} \big| \operatorname{Tr} X^k \big| \boldsymbol{a} \big\rangle_\epsilon \tag{3.12}$$

for any $k \geq 0$. We introduce the microscopic "loop insertion operator"

$$\mathscr{L}_{\mathrm{mic}}(z) = -\frac{1}{z} \frac{\partial}{\partial \lambda_{-1}} - \sum_{k \geq 1} \frac{k}{z^{k+1}} \frac{\partial}{\partial \lambda_{k-1}} \,, \tag{3.13}$$

or equivalently

$$\mathscr{L}''_{\mathrm{mic}}(z) = -N \sum_{k \geq 2} \frac{k}{z^{k+1}} \frac{\partial}{\partial t_{k-1}} \,. \tag{3.14}$$

Note that the derivatives are taken at $\boldsymbol{a}$ fixed here, whereas in the similar macroscopic operator (2.20) the derivatives are at $\boldsymbol{s}$ fixed. This point should be clear and we shall not repeat it in the following. Using (3.11), equations (1.48) and (1.54) are then equivalent to

$$R_{\mathrm{mic}}(z) = \frac{N}{z} + \frac{a}{z^2} - \frac{2}{N} \mathscr{L}''_{\mathrm{mic}}(z) \cdot \mathcal{F}_{\mathrm{mic}} \,, \tag{3.15}$$

$$S_{\mathrm{mic}}(z) = -\mathscr{L}_{\mathrm{mic}}(z) \cdot W_{\mathrm{mic}} \,. \tag{3.16}$$

Applying the operator $\mathscr{L}_{\mathrm{mic}}(z)$ on (1.53) and using (3.16) then yields

$$S_{\mathrm{mic}}(z) = -\frac{1}{2i\pi} \oint_\alpha \mathscr{L}_{\mathrm{mic}}(z) \cdot R_{\mathrm{mic}}(z') W(z') \, \mathrm{d}z' \,. \tag{3.17}$$

On the other hand, if we take the derivative of (1.37) with respect to the couplings $\boldsymbol{t}$ or $\boldsymbol{\lambda}$ using (1.44) and (1.45) and taking into account (3.12), we get

$$\mathscr{L}''_{\mathrm{mic}}(z) \cdot R_{\mathrm{mic}}(z') = -\frac{N}{\epsilon^2} \bigg(\Big\langle \boldsymbol{a} \Big| \operatorname{Tr} \frac{1}{z - X} \operatorname{Tr} \frac{1}{z' - X} \Big| \boldsymbol{a} \Big\rangle_\epsilon$$
$$- \Big\langle \boldsymbol{a} \Big| \operatorname{Tr} \frac{1}{z - X} \Big| \boldsymbol{a} \Big\rangle_\epsilon \Big\langle \boldsymbol{a} \Big| \operatorname{Tr} \frac{1}{z' - X} \Big| \boldsymbol{a} \Big\rangle_\epsilon \bigg) \,. \tag{3.18}$$

Plugging (3.18) into (3.17), we obtain (3.9).

To get a more explicit result, we use

$$\mathscr{L}''_{\mathrm{mic}}(z) \cdot R_{\mathrm{mic}}(z') = \mathscr{L}''_{\mathrm{mic}}(z') \cdot R_{\mathrm{mic}}(z) \,, \tag{3.19}$$

which yields

$$\frac{1}{2i\pi} \oint_\alpha \mathscr{L}''_{\mathrm{mic}}(z) \cdot R_{\mathrm{mic}}(z') W(z') \, \mathrm{d}z' = \frac{1}{2i\pi} \oint_\alpha \mathscr{L}''_{\mathrm{mic}}(z') W(z') \, \mathrm{d}z' \cdot R_{\mathrm{mic}}(z)$$
$$= -N \sum_{k \geq 1} g_k \frac{\partial R_{\mathrm{mic}}(z)}{\partial t_k} \,. \tag{3.20}$$

Equation (3.17) thus implies that

$$S''_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = N \sum_{k \geq 1} g_k \frac{\partial R_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{t})}{\partial t_k}\,. \tag{3.21}$$

This important result shows that $S''_{\mathrm{mic}}(z)$ is a meromorphic function on the curve $\mathcal{C}_{\mathrm{mic}}$ (3.2), since $R_{\mathrm{mic}}(z)$ and thus also its variations have this property. We can actually go one step further. Using (3.21), the derivatives of equations (3.5) and (3.6) with respect to the t_ks indeed imply that

$$\oint_{\alpha_i} S''_{\mathrm{mic}}\,\mathrm{d}z = 0\,, \quad \oint_{\delta_i} S''_{\mathrm{mic}}\,\mathrm{d}z = 0\,. \tag{3.22}$$

This shows that

$$S'_{\mathrm{mic}}(P) = \int_{P_0}^{P} S''_{\mathrm{mic}}\,\mathrm{d}z \tag{3.23}$$

is also a well-defined meromorphic function on $\mathcal{C}_{\mathrm{mic}}$, since (3.22) ensures that $S'_{\mathrm{mic}}(P)$ does not depend on the path chosen to perform the integral in (3.23).

Another way to see this, and to gain more information on the pole structure of S'_{mic}, is to use the microscopic quantum characteristic function

$$F_{\mathrm{mic}}(P) = \langle \boldsymbol{a} | \det(z - X) | \boldsymbol{a} \rangle = \mu_0^N \exp \int_{P_0}^{P} R_{\mathrm{mic}}\,\mathrm{d}z\,. \tag{3.24}$$

It is a single-valued function on $\mathcal{C}_{\mathrm{mic}}$ thank's to (3.5) and (3.6). It is actually holomorphic everywhere except at the points at infinity. It has a pole of order N at P_0 and an essential singularity at Q_0,

$$F_{\mathrm{mic}}(z) \underset{z \to \infty}{\sim} z^N\,, \quad \hat{F}_{\mathrm{mic}}(z) \underset{z \to \infty}{\sim} \frac{e^{NV(z)}}{z^N}\,. \tag{3.25}$$

We have used as usual the hatted notation to indicate the value of a function on the second sheet of the hyperelliptic curve on which it is defined. Let us remark that (3.25) and (2.64) coincide, not surprisingly, but (3.25) is valid for any $\boldsymbol{a}$ whereas (2.64) is valid only at $\boldsymbol{s} = \boldsymbol{s}^*$. The most general form for F_{mic} that is compatible with the above constraints is

$$F_{\mathrm{mic}}(z) = \phi_1(z) + \phi_2(z)y\,, \tag{3.26}$$

for some entire functions ϕ_1 and ϕ_2 on the complex plane whose asymptotics can be found from (3.25). Moreover, it is manifest from the definition (3.24) and the explicit formula (3.3) for R_{mic} that F_{mic} can never vanish at finite z. Thus we also have

$$\frac{1}{F_{\mathrm{mic}}(z)} = \varphi_1(z) + \varphi_2(z)y\,, \tag{3.27}$$

for some entire functions φ_1 and φ_2. Let us now consider objects of the form

$$f_\delta(z) = \frac{\delta F_{\mathrm{mic}}}{F_{\mathrm{mic}}} = \delta \ln F_{\mathrm{mic}}\,, \tag{3.28}$$

where δ represents the derivative with respect to a parameter, for example the t_k or the a_i. Clearly f_δ is a meromorphic function on $\mathcal{C}_{\mathrm{mic}}$, because $\delta \ln \hat{F}_{\mathrm{mic}} \underset{z \to \infty}{\sim} N \delta V(z)$ has at most an ordinary pole at Q_0. Moreover, it follows from (3.2) that

$$\delta y = \frac{\rho_\delta}{y} \tag{3.29}$$

for some polynomial ρ_δ. Putting (3.26), (3.27), (3.28) and (3.29) together, we find that

$$f_\delta(z) = p_\delta(z) + \frac{q_\delta(z)}{y}. \tag{3.30}$$

The entire functions p_δ and q_δ must actually be polynomials because f_δ is meromorphic.

If we apply this remark to the primitive of (3.21), which reads

$$S'_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = N \sum_{k \geq 1} g_k \frac{\partial \ln F_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{t})}{\partial t_k}, \tag{3.31}$$

we deduce that S'_{mic} must take the form

$$S'_{\mathrm{mic}}(z) = \frac{N}{2} \left(p(z) + \frac{E_S(z)}{y} \right) \tag{3.32}$$

for some polynomials p and E_S (the overall factor has been chosen for convenience). If we now use

$$R_{\mathrm{mic}}(z) + \hat{R}_{\mathrm{mic}}(z) = N V'(z), \tag{3.33}$$

that follows from (3.3), and (3.21), we find

$$S''_{\mathrm{mic}}(z) + \hat{S}''_{\mathrm{mic}}(z) = N W'''(z). \tag{3.34}$$

Integrating, we obtain

$$S'_{\mathrm{mic}}(z) + \hat{S}'_{\mathrm{mic}}(z) = N W''(z) + c = N p(z). \tag{3.35}$$

The constant of integration c is found to vanish by looking at the limit $z = \mu_0 \to \infty$ and using

$$\int_{\beta_i} S''_{\mathrm{mic}} \, \mathrm{d}z = N W''(\mu_0), \tag{3.36}$$

which is a straightforward consequence of (3.21) and (3.6). Overall, we have thus obtained

$$S'_{\mathrm{mic}}(z) = \frac{N}{2} \left(W''(z) + \frac{E_S(z)}{y} \right). \tag{3.37}$$

The polynomial E_S is determined in the following way. First, from the asymptotics

$$S'_{\mathrm{mic}}(z) \underset{z \to \infty}{\sim} -\frac{v_{0,\mathrm{mic}}}{z^2}, \tag{3.38}$$

we deduce that $\deg E_S = N + \deg W'' = N + d_W - 1$. The condition (3.38) actually puts $d_W + 1$ constraints on the coefficients of E_S, which leaves $N - 1$ unknown. The missing constraints, to be derived shortly, are given by

$$\oint_{\alpha_i} S'_{\mathrm{mic}} \, \mathrm{d}z = 0. \tag{3.39}$$

This is the microscopic analogue to (2.35). Equations (3.37), (3.38) and (3.39) give a full prescription to compute exactly S'_{mic}, and thus all the generalized glueball operators in the microscopic formalism. In particular, we have checked the result successfully with the explicit calculations made in [2].

To finish, let us prove (3.39). The idea is to integrate by part and then to use (3.21) and (1.52),

$$\oint_{\alpha_i} S'_{\mathrm{mic}}\, \mathrm{d}z = -\oint_{\alpha_i} z S''_{\mathrm{mic}}\, \mathrm{d}z = -N \sum_{k\geq 1} g_k \frac{\partial}{\partial t_k} \oint_{\alpha_i} \lambda_{\mathrm{mic}}\,. \tag{3.40}$$

The result then follows by taking the derivative of (3.7) with respect to t_k, which yields zero. It is natural to ask about the δ_i-periods of S'_{mic}. Clearly, they will *not* vanish in general, and thus S_{mic} is *not* well-defined on the curve (3.2). Actually, it is an infinitely-many valued function of z. This is very different from the macroscopic function S_{mac}, which is always two-valued because it satisfies the anomaly equation (2.7). For the equality $S^*_{\mathrm{mac}} = S^*_{\mathrm{mic}}$ to be valid, the microscopic equations of motion (1.57) must put constraints on the β_i-periods of $S'_{\mathrm{mic}}\mathrm{d}z$. This is what we are going to study now.

3.3 The microscopic quantum equations of motion

We define, in strict parallel with what was done in section 2.2.1 for the curve (2.13), a canonical basis of one-forms $\{h_i\}_{1\leq i\leq N}$ on $\mathcal{C}_{\mathrm{mic}}$,

$$\frac{1}{2i\pi} \oint_{\alpha_j} h_i = \delta_{ij}\,, \tag{3.41}$$

holomorphic everywhere except at the points at infinity where they may have simple poles. Explicitly,

$$h_i = \psi_i\, \mathrm{d}z = \frac{p_i}{y}\, \mathrm{d}z\,, \tag{3.42}$$

where the $p_i(z) = z^{N-1} + \cdots$ are monic polynomials of degree $N-1$ fixed by the conditions (3.41). We shall need the identity

$$\frac{\partial \lambda_{\mathrm{mic}}}{\partial a_i} = h_i - \mathrm{d}(z\psi_i)\,. \tag{3.43}$$

To derive this equation, let us write

$$\lambda_{\mathrm{mic}} = z R_{\mathrm{mic}}\, \mathrm{d}z = -\ln F_{\mathrm{mic}}\, \mathrm{d}z + \mathrm{d}(z\ln F_{\mathrm{mic}})\,. \tag{3.44}$$

From (3.28) and (3.30) applied to $\delta = \partial/\partial a_i$, we know that

$$\frac{\partial \ln F_{\mathrm{mic}}}{\partial a_i} = p(z) + \frac{q(z)}{y} \tag{3.45}$$

for some polynomials p and q. Integrating (3.33) with respect to z and then taking the derivative with respect to a_i, we find that necessarily $p = 0$. Moreover,

$$\frac{1}{F_{\mathrm{mic}}} \frac{\partial F_{\mathrm{mic}}}{\partial a_i} = \mathcal{O}(1/z)\,, \tag{3.46}$$

and thus $\deg q = N - 1$. This shows that $\partial \ln F_{\text{mic}} / \partial a_i$ is a linear combination of the ψ_i. Comparing (3.41) with the derivative of (3.7) with respect to a_j, we actually find

$$\frac{\partial \ln F_{\text{mic}}}{\partial a_i} = -\psi_i \qquad (3.47)$$

which, using (3.44), is equivalent to (3.43).

The definition (1.53) yields

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = \frac{1}{2i\pi} \oint_\alpha \frac{W}{z} \frac{\partial \lambda_{\text{mic}}}{\partial a_i} \,. \qquad (3.48)$$

Using (3.43) and an integration by part, we thus obtain

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = \frac{1}{2i\pi} \oint_\alpha W' h_i \,. \qquad (3.49)$$

We now apply the Riemann bilinear relation (A.12) with $F = S'_{\text{mic}}$ and $g = h_i$. The calculation is strictly similar to the one done in section 2.3.2, and thus we shall be brief. Using (3.39), we get, using the same definition as in (2.49),

$$\begin{aligned}
\int_{\beta_i} S'_{\text{mic}} \, \mathrm{d}z &= -\frac{1}{2i\pi} \oint_\alpha \left[S'_{\text{mic}} H_i + \left(N W'' - S'_{\text{mic}} \right) \left(\int_{\beta_r} h_i + 2\ln\mu_0 - H_i \right) \right] \\
&= N W'(\mu_0) - \frac{N}{2i\pi} \oint_\alpha W' h_i \,.
\end{aligned} \qquad (3.50)$$

Equation (3.49) is thus equivalent to

$$\frac{\partial W_{\text{mic}}}{\partial a_i} = -\frac{1}{N} \int_{\beta_i} S'_{\text{mic}} \, \mathrm{d}z + W'(\mu_0) \,. \qquad (3.51)$$

This is exactly what the discussion at the end of 3.2 was suggesting. On-shell, we have

$$\int_{\beta_i} S^{*\prime}_{\text{mic}} \, \mathrm{d}z = N W'(\mu_0) \,. \qquad (3.52)$$

In particular, $\oint_{\delta_i} S^{*\prime}_{\text{mic}} \mathrm{d}z = 0$ which, together with (3.39), implies that S^*_{mic} is a meromorphic function on (3.2).

It is important to note that the curve (3.2) can degenerate (and actually must always do so if $\deg W' < N$) on the solutions to (3.51). A very explicit discussion of how this happens is given in [1]. To understand better this point, let us look in more details into the consequences of (3.52). Since S^*_{mic} is two-sheeted, we can integrate (3.35) (remember that (3.36) implies that $c = 0$) to get

$$S^*_{\text{mic}}(z) + \hat{S}^*_{\text{mic}}(z) = N W'(z) + \tilde{c} \,. \qquad (3.53)$$

Looking at the limit $z = \mu_0 \to \infty$, we obtain

$$\int_{\beta_i} S^{*\prime}_{\text{mic}} \, \mathrm{d}z = N W'(\mu_0) + \tilde{c} \qquad (3.54)$$

and thus the equation of motion (3.52) implies that $\tilde{c} = 0$. The function $S^*_{\text{mic}}(NW' - S^*_{\text{mic}})$ is thus single-valued and meromorphic, meaning that it must be a rational function of z. Poles for finite values of z at points where z is a good coordinate on the curve (i.e. excluding the branching points) would yield similar poles for $S^{*\prime}_{\text{mic}}$ which we know are not present, see (3.37). Similarly, a pole of order n at a branching point yields after taking the derivative with respect to z a pole of order $n + 2$ for $S^{*\prime}_{\text{mic}}$ (because $\mathrm{d}y/\mathrm{d}z \propto 1/y$). This is impossible because (3.37) shows that $S^{*\prime}_{\text{mic}}$ only has simple poles at the branching points. Finally, this discussion implies that

$$S^*_{\text{mic}}(z)\big(NW'(z) - S^*_{\text{mic}}(z)\big) = N^2 \Delta^*_{S,\,\text{mic}}(z) \tag{3.55}$$

is a polynomial whose degree is fixed to be $d_W - 1$ by the asymptotics $S^*_{\text{mic}} = \mathcal{O}(1/z)$ at infinity. Solving this quadratic equation we find that

$$S^*_{\text{mic}}(z; \boldsymbol{g}, \boldsymbol{t}) = \frac{N}{2}\Big(W'(z) - \sqrt{W'(z)^2 - 4\Delta^*_{S,\,\text{mic}}(z)}\Big). \tag{3.56}$$

Consistency with the fact that S^*_{mic} is also defined on (3.2) implies that the following factorization conditions must be satisfied,

$$y^2 = M_{N-r}(z)^2 y^2_{\text{mic},\,r}\,, \tag{3.57}$$

$$W'(z)^2 - 4\Delta^*_{S,\,\text{mic}}(z) = N_{d_W-r}(z)^2 y^2_{\text{mic},\,r}\,, \tag{3.58}$$

for some polynomials M_{N-r} and N_{d_W-r} of degrees $N - r$ and $d_W - r$ respectively. A priori, the integer r can take any value consistent with the above equations. We see that both S^*_{mic} and R^*_{mic} are defined on a genus $r - 1$ reduced curve

$$\mathcal{C}_{\text{mic},\,r}:\ y^2_{\text{mic},\,r} = \prod_{i=1}^{r}(z - v_i^-)(z - v_i^+) \tag{3.59}$$

which is obtained from (3.2) by joining some of the branch cuts. For example, if the ith and j^{th} branch cuts join, then the resulting cut has a β-type contour $\beta'_i = \beta_i = \beta_j$ and a α-type contour $\alpha'_i = \alpha_i + \alpha_j$. Reshuffling the indices appropriately and renaming the contours α_i and β_i on the reduced curve to match the notations used in the macroscopic formalism, we see that (3.5) and (3.6) become

$$\oint_{\alpha_i} R^*_{\text{mic}}\,\mathrm{d}z \in 2i\pi\mathbb{Z}\,, \tag{3.60}$$

$$\int_{\beta_i} R^*_{\text{mic}}\,\mathrm{d}z - NV(\mu_0) + 2N\ln\mu_0 \in 2i\pi\mathbb{Z}\,, \tag{3.61}$$

on the reduced curve. The integer appearing on the right hand side of (3.60) is simply given by the number of cuts of the original curve $\mathcal{C}_{\text{mic}}$ that have joined to form the ith cut of the reduced curve $\mathcal{C}_{\text{mic},\,r}$.

The link with the macroscopic formalism is almost complete. To finish the proof, we need to show that R^*_{mic} and S^*_{mic} are related to each other consistently with the anomaly equation (2.6) that relates R^*_{mac} and S^*_{mac} (note that (3.55) already shows that S^*_{mic} satisfies an equation like (2.7)). We could do that by carefully analyzing the properties of R^*_{mic} and S^*_{mic}, but the most elegant route is to perform a full microscopic analysis of the anomaly equations. This is the subject of the next section.

3.4 The microscopic derivation of the anomaly equations

At the perturbative level, the anomaly equations are generated by the operators (2.3). Their action on the chiral operators is given by (2.4) and their algebra by (2.5). As discussed at length in [2], these operators must undergo very strong non-perturbative quantum corrections. In particular, at the non-perturbative level, $L_n \cdot u_m$, $J_n \cdot u_m$, $L_n \cdot v_m$ and $J_n \cdot v_m$ are not single-valued functions of the u_ps and v_qs, and moreover the algebra generated by L_n and J_m does not close.

In [2], it was conjectured that the non-perturbative operators are given by[7]

$$L_n = -\frac{1}{2i\pi} \sum_{i=1}^{N} \oint_{\alpha_i} z^{n+1} R_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{t}) \, \mathrm{d}z \, \frac{\partial}{\partial a_i} \,, \tag{3.62}$$

$$J_n = -\frac{1}{2i\pi} \sum_{i=1}^{N} \oint_{\alpha_i} z^{n+1} S_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \, \mathrm{d}z \, \frac{\partial}{\partial a_i} \,. \tag{3.63}$$

The fundamental requirement is that they generate the correct anomaly polynomials

$$N L_n \cdot W_{\mathrm{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = \mathscr{A}_n(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \,, \tag{3.64}$$

$$N J_n \cdot W_{\mathrm{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) = \mathscr{B}_n(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \,, \tag{3.65}$$

with

$$\mathscr{A}_n = -N \sum_{k \geq 0} g_k u_{n+k+1, \, \mathrm{mic}} - N \sum_{k \geq 0} \lambda_k v_{n+k+1, \, \mathrm{mic}} + 2 \sum_{k_1 + k_2 = n} u_{k_1, \, \mathrm{mic}} v_{k_2, \, \mathrm{mic}} \,, \tag{3.66}$$

$$\mathscr{B}_n = -N \sum_{k \geq 0} g_k v_{n+k+1, \, \mathrm{mic}} + \sum_{k_1 + k_2 = n} v_{k_1, \, \mathrm{mic}} v_{k_2, \, \mathrm{mic}} \,. \tag{3.67}$$

Equivalently, in terms of the generating function we have

$$\begin{aligned}
\mathscr{A}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) &= \sum_{n \geq -1} \frac{\mathscr{A}_n(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t})}{z^{n+2}} = N L(z) \cdot W_{\mathrm{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \\
&= -N W'(z; \boldsymbol{g}) R_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{t}) - N V'(z; \boldsymbol{t}) S_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \\
&\quad + 2 R_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{t}) S_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) + N^2 \Delta_{R, \, \mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \,,
\end{aligned} \tag{3.68}$$

$$\begin{aligned}
\mathscr{B}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) &= \sum_{n \geq -1} \frac{\mathscr{B}_n(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t})}{z^{n+2}} = N J(z) \cdot W_{\mathrm{mic}}(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \\
&= -N W'(z; \boldsymbol{g}) S_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \\
&\quad + S_{\mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t})^2 + N^2 \Delta_{S, \, \mathrm{mic}}(z; \boldsymbol{a}, \boldsymbol{g}, \boldsymbol{t}) \,,
\end{aligned} \tag{3.69}$$

for some polynomials $\Delta_{R, \, \mathrm{mic}}(z)$ and $\Delta_{S, \, \mathrm{mic}}(z)$. Equations (3.64) and (3.65) were checked in [2] up to two instantons. Using the technology developed in the previous sections, we can now prove these equations independently of the small q expansion.

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[7]In [2], only the usual theory with V constant was considered, but the formulas for L_n and J_n straightforwardly generalize to the case of arbitrary $\boldsymbol{t}$.

3.4.1 The action of L_n

We use (3.62) and (3.51) to write

$$NL_n \cdot W_{\mathrm{mic}} = \frac{1}{2i\pi} \sum_i \oint_{\alpha_i} z^{n+1} R_{\mathrm{mic}} \, \mathrm{d}z \int_{\beta_i} S'_{\mathrm{mic}} \, \mathrm{d}z - NW'(\mu_0) u_{n+1,\,\mathrm{mic}} \,. \tag{3.70}$$

The right-hand side of this equation can be evaluated by using the Riemann bilinear relation (A.12), with $p = z^{n+1}$, $F = R_{\mathrm{mic}}$ and $G = S_{\mathrm{mic}}$. Using (3.33),

$$S_{\mathrm{mic}}(z) + \hat{S}_{\mathrm{mic}}(z) = NW'(z) + \int_{\beta_r} S'_{\mathrm{mic}} \, \mathrm{d}z - NW'(\mu_0) \,, \tag{3.71}$$

which is obtained from (3.35) in a way similar to (2.61), and (3.39) we get

$$NL_n \cdot W_{\mathrm{mic}} = - NW'(\mu_0) u_{n+1,\,\mathrm{mic}} + u_{n+1,\,\mathrm{mic}} \int_{\beta_r} S'_{\mathrm{mic}} \, \mathrm{d}z \tag{3.72}$$

$$+ \frac{1}{2i\pi} \oint_\alpha z^{n+1} \left[R_{\mathrm{mic}} S_{\mathrm{mic}} + \left(NV' - R_{\mathrm{mic}} \right) \left(NW' - S_{\mathrm{mic}} - NW'(\mu_0) + \int_{\beta_r} S'_{\mathrm{mic}} \mathrm{d}z \right) \right]$$

$$= \frac{1}{2i\pi} \oint_\alpha z^{n+1} \left(-NW' R_{\mathrm{mic}} - NV' S_{\mathrm{mic}} + 2 R_{\mathrm{mic}} S_{\mathrm{mic}} \right) \mathrm{d}z = \mathscr{A}_n \,.$$

We have thus derived the first anomaly equation (3.64).

3.4.2 The action of J_n

The definition (3.63) yields

$$NJ_n \cdot W_{\mathrm{mic}} = \frac{1}{2i\pi} \sum_i \oint_{\alpha_i} z^{n+1} S_{\mathrm{mic}} \, \mathrm{d}z \int_{\beta_i} S'_{\mathrm{mic}} \, \mathrm{d}z - NW'(\mu_0) v_{n+1,\,\mathrm{mic}} \,. \tag{3.73}$$

To perform the calculation, we apply the Riemann bilinear relation with $F = G = S_{\mathrm{mic}}$. In this case, the full power of the generalized relation (A.10) derived in the appendix is needed, because $F = S_{\mathrm{mic}}$ is multi-valued on the curve (3.2) as explained at the end of section 3.2. Because $F = G$, the net effect of the additional terms in (A.10) with respect to the more conventional formula (A.12) is a crucial global factor of 2,

$$2 \times \frac{1}{2i\pi} \sum_i \oint_{\alpha_i} z^{n+1} S_{\mathrm{mic}} \, \mathrm{d}z \int_{\beta_i} S'_{\mathrm{mic}} \, \mathrm{d}z = 2 \times v_{n+1,\,\mathrm{mic}} \int_{\beta_r} S'_{\mathrm{mic}} \, \mathrm{d}z$$

$$+ \frac{1}{2i\pi} \oint_\alpha z^{n+1} \left[S^2_{\mathrm{mic}} + \left(NW' - S_{\mathrm{mic}} - NW'(\mu_0) + \int_{\beta_r} S'_{\mathrm{mic}} \, \mathrm{d}z \right)^2 \right] \tag{3.74}$$

$$= \frac{1}{i\pi} \oint_\alpha z^{n+1} \left(-NW' S_{\mathrm{mic}} + S^2_{\mathrm{mic}} \right) \mathrm{d}z + 2 NW'(\mu_0) v_{n+1,\,\mathrm{mic}} \,.$$

Plugging this result in (3.73), we obtain (3.65) as we wished, completing the full microscopic derivation of the generalized Konishi anomaly equations.

3.5 The solution in the microscopic formalism

When the quantum equations of motion (1.57) are satisfied, we have automatically

$$L_n \cdot W_{\mathrm{mic}} = 0\,, \quad J_n \cdot W_{\mathrm{mic}} = 0\,, \tag{3.75}$$

and thus R^*_{mic} and S^*_{mic} satisfy the anomaly equations. We can thus summarize our findings as follows. The functions R^*_{mic} and S^*_{mic} are fully determined by the constraints

$$-NW'(z)R^*_{\mathrm{mic}}(z) - NV'(z)S^*_{\mathrm{mic}}(z) + 2R^*_{\mathrm{mic}}(z)S^*_{\mathrm{mic}}(z) + N^2\Delta_{R,\,\mathrm{mic}}(z) = 0\,,$$
$$-NW'(z)S^*_{\mathrm{mic}}(z) + S^*_{\mathrm{mic}}(z)^2 + N^2\Delta_{S,\,\mathrm{mic}}(z) = 0$$
$$R^*_{\mathrm{mic}}(z) \underset{z\to\infty}{\sim} \frac{N}{z}\,, \quad S^*_{\mathrm{mic}}(z) \underset{z\to\infty}{\sim} \frac{v_{0,\,\mathrm{mic}}}{z}\,, \tag{3.76}$$
$$\oint_{\alpha_i} R^*_{\mathrm{mic}}\,\mathrm{d}z \in 2i\pi\mathbb{Z}\,, \quad \int_{\beta_i} R^*_{\mathrm{mic}}\,\mathrm{d}z - NV(\mu_0) + 2N\ln\mu_0 \in 2i\pi\mathbb{Z}\,.$$

The first two constraints are the anomaly equations that we have just derived. They are valid only on-shell, which is unlike their macroscopic counterparts (2.6) and (2.7) which are valid off-shell. The constraints in the third line of (3.76) are the usual asymptotics that follow from the definitions of R_{mic} and S_{mic}. The last constraints in the fourth line of (3.76) correspond to (3.60) and (3.61). The solution is then fixed up to the usual discrete ambiguity corresponding to the existence of a discrete set of vacua.

The *on-shell* solutions of the macroscopic formalism (2.63) and of the microscopic formalism (3.76) are clearly identical. The fundamental result (1.62) is thus proven.

4. Conclusions

Generalizing the work of Nekrasov and collaborators [3, 6, 17] from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, we have provided a microscopic derivation, from first principles, of the exact results in supersymmetric gauge theories, following the ideas explained in [1]. The main highlight is to show that the microscopic approach based on the Nekrasov's sums over colored partitions and the macroscopic approach based on the Dijkgraaf-Vafa matrix model are equivalent. A particularly interesting application is the non-perturbative derivation of the generalized Konishi anomaly equations.

The macroscopic and microscopic formalisms have striking structural similarities. This is illustrated in tables 2 and 3, where each equation in one formalism is associated to a similar equation in the other formalism.

A basic property of the correspondence between the formalisms is that an off-shell, or "kinematical" relation on one side is typically only valid on-shell, or "dynamically" on the other side. When both sides are put on-shell, they yield equivalent results. This is reminiscent of the electric/magnetic duality, which exchanges Bianchi identities with equations of motion, and seems to be a common feature of many non-trivial dualities. Actually, the duality we have been discussing is directly related to the open/closed string duality, the open string description corresponding to the microscopic formalism and the closed string description to the macroscopic formalism.

Macroscopic formalism	Microscopic formalism
$v_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g}) = N\varepsilon\langle\!\langle\boldsymbol{s}|\,\mathrm{Tr}\,X^k|\boldsymbol{s}\rangle\!\rangle$	$u_{k,\,\mathrm{mic}}(\boldsymbol{a},\boldsymbol{t}) = \langle\boldsymbol{a}|\,\mathrm{Tr}\,X^k|\boldsymbol{a}\rangle$
$\lambda_{\mathrm{mac}} = S_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g})\,\mathrm{d}z$	$\lambda_{\mathrm{mic}} = zR_{\mathrm{mic}}(z;\boldsymbol{a},\boldsymbol{t})\,\mathrm{d}z$
$\boldsymbol{s} = \dfrac{1}{2i\pi N}\oint_{\alpha}\lambda_{\mathrm{mac}}$	$\boldsymbol{a} = \dfrac{1}{2i\pi}\oint_{\alpha}\lambda_{\mathrm{mic}}$
$\dfrac{\partial\lambda_{\mathrm{mac}}}{\partial s_i} = Nh_i$	$\dfrac{\partial\lambda_{\mathrm{mic}}}{\partial a_i} = h_i - \mathrm{d}(z\psi_i)$
$W_{\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g},\boldsymbol{t}) = \dfrac{1}{2i\pi}\oint_{\alpha}S_{\mathrm{mac}}V\,\mathrm{d}z$ $- \sum_i N_i\dfrac{\partial\mathcal{F}_{\mathrm{mac}}}{\partial s_i}$	$W_{\mathrm{mic}}(\boldsymbol{a},\boldsymbol{g},\boldsymbol{t}) = \dfrac{1}{2i\pi}\oint_{\alpha}R_{\mathrm{mic}}W\,\mathrm{d}z$
$\sum_i s_i\dfrac{\partial W_{\mathrm{mac}}}{\partial s_i} =$ $-\dfrac{1}{2i\pi}\oint_{\alpha}R_{\mathrm{mac}}W\,\mathrm{d}z + W_{\mathrm{mac}}$	$\sum_i a_i\dfrac{\partial W_{\mathrm{mac}}}{\partial a_i} = -2v_{0,\,\mathrm{mic}}$ $+\dfrac{1}{2i\pi}\oint_{\alpha}\left(zW'R_{\mathrm{mic}} + zV'S_{\mathrm{mic}}\right)\mathrm{d}z$
$u_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g},\boldsymbol{t}) = k\dfrac{\partial W_{\mathrm{mac}}}{\partial g_{k-1}}$	$v_{k,\,\mathrm{mic}}(\boldsymbol{a},\boldsymbol{g},\boldsymbol{t}) = \dfrac{N}{k+1}\dfrac{\partial W_{\mathrm{mic}}}{\partial t_{k+1}}$
$v_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g}) = \dfrac{N}{k+1}\dfrac{\partial W_{\mathrm{mac}}}{\partial t_{k+1}}$	$u_{k,\,\mathrm{mic}}(\boldsymbol{a},\boldsymbol{t}) = k\dfrac{\partial W_{\mathrm{mic}}}{\partial g_{k-1}}$
$v_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g}) = -Nk\dfrac{\partial\mathcal{F}_{\mathrm{mac}}}{\partial g_{k-1}}\,,\ k\geq 1$	$u_{k,\,\mathrm{mic}}(\boldsymbol{a},\boldsymbol{t}) = 2k\dfrac{\partial\mathcal{F}_{\mathrm{mic}}}{\partial t_{k-1}}\,,\ k\geq 2$
$u_{k,\,\mathrm{mac}}(\boldsymbol{s},\boldsymbol{g},\boldsymbol{t}) = \dfrac{1}{2i\pi}\oint_{\alpha}z^k\sum_i N_i h_i$ $-\dfrac{N}{\varepsilon^2}\Big[\langle\!\langle\boldsymbol{s}|\varepsilon\mathrm{Tr}\,X^k\,\varepsilon\mathrm{Tr}\,V(X)|\boldsymbol{s}\rangle\!\rangle_{\varepsilon}$ $-\langle\!\langle\boldsymbol{s}|\varepsilon\mathrm{Tr}\,X^k|\boldsymbol{s}\rangle\!\rangle_{\varepsilon}\langle\!\langle\boldsymbol{s}|\varepsilon\mathrm{Tr}\,V(X)|\boldsymbol{s}\rangle\!\rangle_{\varepsilon}\Big]$	$v_{k,\,\mathrm{mic}}(\boldsymbol{a},\boldsymbol{t}) = \dfrac{N}{(k+1)(k+2)}\times$ $\dfrac{1}{\epsilon^2}\Big[\langle\boldsymbol{a}|\,\mathrm{Tr}\,X^{k+2}\,\mathrm{Tr}\,W(X)|\boldsymbol{a}\rangle_{\epsilon}$ $-\langle\boldsymbol{a}|\,\mathrm{Tr}\,X^{k+2}|\boldsymbol{a}\rangle_{\epsilon}\langle\boldsymbol{a}|\,\mathrm{Tr}\,W(X)|\boldsymbol{a}\rangle_{\epsilon}\Big]$
$R_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g},\boldsymbol{t}) = \sum_i N_i\psi_i(z)$ $+\sum_{k\geq 0}\lambda_k\dfrac{\partial S_{\mathrm{mac}}(z;\boldsymbol{s},\boldsymbol{g})}{\partial g_k}$	$S''_{\mathrm{mic}}(z;\boldsymbol{a},\boldsymbol{g},\boldsymbol{t}) =$ $N\sum_{k\geq 1}g_k\dfrac{\partial R_{\mathrm{mic}}(z;\boldsymbol{a},\boldsymbol{t})}{\partial t_k}$
$\dfrac{1}{2i\pi}\oint_{\alpha_i}S'_{\mathrm{mac}}\,\mathrm{d}z = 0$	$\dfrac{1}{2i\pi}\oint_{\alpha_i}R_{\mathrm{mic}}\,\mathrm{d}z \in \mathbb{Z}$
$\displaystyle\int_{\beta_i}S'_{\mathrm{mac}}\,\mathrm{d}z = NW'(\mu_0)$	$\displaystyle\int_{\beta_i}R_{\mathrm{mic}}\,\mathrm{d}z = NV(\mu_0) - 2N\ln\mu_0$

Table 2: Detailed formal mapping between the macroscopic and microscopic formalisms.

There are many ways to generalize the present work. Each particular $\mathcal{N} = 1$ gauge theory, with given gauge group and matter content, can be studied along the lines of

Macroscopic formalism	Microscopic formalism
$\dfrac{1}{2i\pi}\displaystyle\oint_{\alpha_i} R_{\mathrm{mac}}\,\mathrm{d}z \in \mathbb{Z}$	$\dfrac{1}{2i\pi}\displaystyle\oint_{\alpha_i} S'_{\mathrm{mic}}\,\mathrm{d}z = 0$
$\dfrac{\partial W_{\mathrm{mac}}}{\partial s_i} = -\displaystyle\int_{\beta_i} R_{\mathrm{mac}}\,\mathrm{d}z$ $\quad - 2N\ln\mu_0 + NV(\mu_0)$	$\dfrac{\partial W_{\mathrm{mic}}}{\partial a_i} = -\dfrac{1}{N}\displaystyle\int_{\beta_i} S'_{\mathrm{mic}}\,\mathrm{d}z + W'(\mu_0)$
$-NW'R_{\mathrm{mac}} - NV'S_{\mathrm{mac}} + 2R_{\mathrm{mac}}S_{\mathrm{mac}}$ $\quad + N^2\Delta_R = 0$	$NL(z)\cdot W_{\mathrm{mic}} = -NW'R_{\mathrm{mic}}$ $\quad - NV'S_{\mathrm{mic}} + 2R_{\mathrm{mic}}S_{\mathrm{mic}} + N^2\Delta_{R,\,\mathrm{mic}}$
$-NW'S_{\mathrm{mac}} + S^2_{\mathrm{mac}} + N^2\Delta_S = 0$	$NJ(z)\cdot W_{\mathrm{mic}} = -NW'S_{\mathrm{mic}}$ $\quad + S^2_{\mathrm{mic}} + N^2\Delta_{S,\,\mathrm{mic}}$
$\boldsymbol{R^*_{\mathrm{mac}} = R\,,\quad S^*_{\mathrm{mac}} = S}$	$\boldsymbol{R^*_{\mathrm{mic}} = R\,,\quad S^*_{\mathrm{mic}} = S}$

Table 3: Detailed formal mapping between the macroscopic and microscopic formalisms (follow-up to table 2).

our work, and a nice equivalence between the associated macroscopic and microscopic formalisms should follow.

A particularly interesting avenue of research is to consider deformations of the ordinary gauge theories, by turning on various backgrounds. For example, we can study the theory in a non-zero Ω-background. Many "microscopic" formulas straightforwardly generalize to this case. In particular, the microscopic superpotential is still given by (1.53), because the parameter ϵ is not charged under the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry. *The duality discussed in the present work must generalize to the deformed theory.* In particular, there should exist a deformation of the macroscopic formalism corresponding to turning on ϵ, and the on-shell equivalence between the formalisms should hold to any order in ϵ.[8] Another interesting deformation is the gravitational background discussed in [8], which corresponds to the non-zero parameter ε on the macroscopic side. This deformation has not been studied yet on the microscopic side. Understanding fully these extended dualities is likely to involve interesting physics and mathematics. They are highly non-trivial, yet probably fully solvable, examples of open/closed string dualities.

Acknowledgments

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[8]Actually, the most general Ω-background is characterized by two complex parameters ϵ_1 and ϵ_2, and we have studied the special case $\epsilon = \epsilon_1 = \epsilon_2$. The duality should extend to the most general case as well.

A. Generalized Riemann bilinear relations

In this appendix, we derive an interesting generalization of the Riemann bilinear relations that we have used again and again in the main text.

We consider a hyperelliptic curve

$$\mathcal{C}: \ y^2 = \prod_{i=1}^{r}(z - z_i^-)(z - z_i^+) \tag{A.1}$$

with two marked points P_0 and Q_0 corresponding to $z = \infty$. This curve, which can be conveniently represented as a polygon with suitable identifications on the boundary, is depicted in figure 2. The contours α_i can be chosen to encircle the branch cuts $[z_i^-, z_i^+]$, and the contours β_i join P_0 to Q_0 by going through $[z_i^-, z_i^+]$. The contours δ_i are then defined by $\delta_i = \beta_i - \beta_r$ for $1 \le i \le r - 1$, and we note $\alpha = \sum_i \alpha_i$ the contour at infinity. It is convenient to introduce a regulator μ_0. In formulas containing μ_0, it is always understood that the points P_0 and Q_0 correspond to $(y = \mu_0^r, z = \mu_0)$ and $(y = -\mu_0^r, z = \mu_0)$ respectively and that the limit $\mu_0 \to \infty$ must be taken. If h is a meromorphic function on $\mathcal{C}$, we denote by $h(z)$ its value on the first sheet (which we choose to be the sheet containing P_0) and $\hat{h}(z)$ its value on the second sheet.

We consider two meromorphic functions f and g on $\mathcal{C}$ that are holomorphic everywhere except at the points at infinity where they may have poles of arbitrary order (our discussion can be straightforwardly generalized when poles at finite z are present, but this is not needed for the applications in the main text). Let us first assume that

$$\oint_{\alpha_i} f \, \mathrm{d}z = 0, \quad \oint_{\delta_i} f \, \mathrm{d}z = 0. \tag{A.2}$$

Let us choose a base point O on $\mathcal{C}$, distinct from P_0 or Q_0. A primitive F of f is defined by

$$F(P) = \int_O^P f \, \mathrm{d}z. \tag{A.3}$$

The conditions (A.2) ensure that $F(P)$ does not depend on the path from O to P chosen to perform the integral in (A.3). This means that F is single-valued on the curve $\mathcal{C}$. In particular, we can talk about the values $F(z)$ and $\hat{F}(z)$. It is actually a meromorphic function on $\mathcal{C} \setminus \{P_0, Q_0\}$. If $f \mathrm{d}z$ has a non-zero residue f_0 at P_0 (and thus also a non-zero residue $-f_0$ at Q_0), then F has a logarithmic branch cut running from P_0 to Q_0 across which it jumps by $2i\pi f_0$. We shall always choose this branch cut to go along the contour β_r.

Let us now waive the hypothesis (A.2). The function F is then no longer single-valued on $\mathcal{C}$ and we need to specify the contour from O to P in (A.3). We shall always choose this contour to lie entirely in the interior of the polygon of figure 2, never going to the boundary. If z is fixed, F can then take two values $F(z)$ and $\hat{F}(z)$ in the polygon, modulo the $2i\pi f_0$
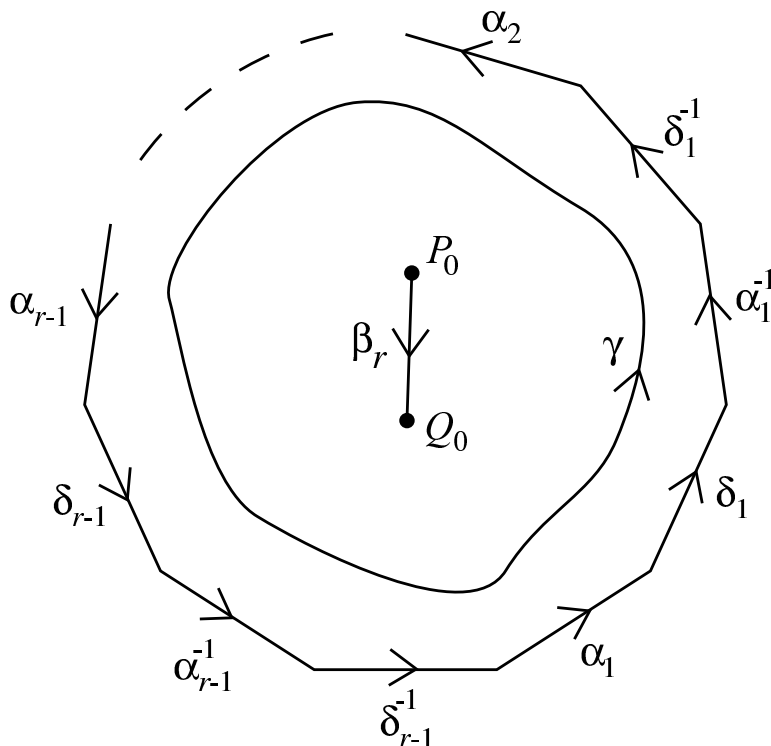
Figure 2: The representation of the surface (A.1) as a $4(r-1)$-gon with suitable identifications on the boundary. We have depicted the contours α_i, $\delta_i = \beta_i - \beta_r$, β_r and γ, as well as the two marked points at infinity P_0 and Q_0.

ambiguity due to the possible logarithmic cut. In the description involving the two sheets of the curve (A.1), $F(z)$ correspond to the value obtained by doing the integral (A.3) following a contour that never circles around a branch cut $[z_i^-, z_i^+]$, whereas the value $\hat{F}(z)$ is obtained by doing the analytic continuation following straight a path that goes through the same branch cut as β_r.

Let us also define the primitive G of g by

$$G(P) = \int_O^P g \, dz \,, \tag{A.4}$$

following exactly the same procedure as for F. We then consider the integral

$$\mathscr{I} = \oint_\gamma pFG \, dz \,, \tag{A.5}$$

where $p(z)$ is an arbitrary polynomial and the contour γ is defined in figure 2. We could of course absorb the polynomial p by redefining F or G, but it is convenient to present the results in this form for our purposes. Let us emphasize that we do *not* assume that relations like (A.2) hold for f or for g. We are going to compute this integral in two different ways, and this will yield the generalization of the Riemann bilinear relations that we are seeking.

Let us first deform the contour γ so that it merges with the boundary of the polygon. If the function FG were single valued on the curve $\mathcal{C}$, we would automatically find zero
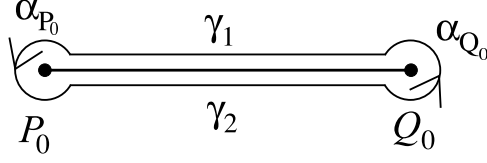
Figure 3: Decomposition of the contour $\gamma = \gamma_1 + \alpha_{P_0} + \gamma_2 + \alpha_{Q_0}$. The contours α_{P_0} and α_{Q_0} are small circles around the points at infinity. The contours γ_1 and γ_2 go along the regularized contour β_r, joining the points $z = \mu_0$ on the first and second sheets of the curve $\mathcal{C}$.

due to the cancellation between the terms coresponding to the integral over α_i and then α_i^{-1}, and over δ_i and then δ_i^{-1}. However, to go from the contour α_i to the contour α_i^{-1}, we have to follow δ_i, which induces a discontinuity $\oint_{\delta_i} f \mathrm{d}z$ for F and $\oint_{\delta_i} g \mathrm{d}z$ for G. The same phenomenon occurs when we go from the contour δ_i to δ_i^{-1}: following α_i^{-1}, we pick discontinuities $- \oint_{\alpha_i} f \mathrm{d}z$ and $- \oint_{\alpha_i} g \mathrm{d}z$ for F and G respectively. Overall we thus obtain

$$
\begin{aligned}
\mathscr{I} = \sum_{i=1}^{r-1} \Bigg[& \oint_{\alpha_i} \left(pFG - p\Big(F + \oint_{\delta_i} f \mathrm{d}u\Big)\Big(G + \oint_{\delta_i} g \mathrm{d}u\Big) \right) \mathrm{d}z \\
& + \oint_{\delta_i} \left(pFG - p\Big(F - \oint_{\alpha_i} f \mathrm{d}u\Big)\Big(G - \oint_{\alpha_i} g \mathrm{d}u\Big) \right) \mathrm{d}z \Bigg] \\
= \sum_{i=1}^{r-1} \Bigg[& \oint_{\delta_i} pF \mathrm{d}z \oint_{\alpha_i} g \mathrm{d}z + \oint_{\delta_i} pG \mathrm{d}z \oint_{\alpha_i} f \mathrm{d}z \\
& - \oint_{\delta_i} g \mathrm{d}z \oint_{\alpha_i} pF \mathrm{d}z - \oint_{\delta_i} f \mathrm{d}z \oint_{\alpha_i} pG \mathrm{d}z \Bigg].
\end{aligned}
\tag{A.6}
$$

Using $\delta_i = \beta_i - \beta_r$ and $\alpha = \sum_i \alpha_i$, we can rewrite the above formula as

$$
\begin{aligned}
\mathscr{I} = \sum_{i=1}^{r} \Bigg[& \int_{\beta_i} pF \mathrm{d}z \oint_{\alpha_i} g \mathrm{d}z + \int_{\beta_i} pG \mathrm{d}z \oint_{\alpha_i} f \mathrm{d}z - \int_{\beta_i} g \mathrm{d}z \oint_{\alpha_i} pF \mathrm{d}z - \int_{\beta_i} f \mathrm{d}z \oint_{\alpha_i} pG \mathrm{d}z \Bigg] \\
& - \int_{\beta_r} pF \mathrm{d}z \oint_{\alpha} g \mathrm{d}z - \int_{\beta_r} pG \mathrm{d}z \oint_{\alpha} f \mathrm{d}z + \int_{\beta_r} g \mathrm{d}z \oint_{\alpha} pF \mathrm{d}z + \int_{\beta_r} f \mathrm{d}z \oint_{\alpha} pG \mathrm{d}z.
\end{aligned}
\tag{A.7}
$$

A second way to compute $\mathscr{I}$ is to deform the contour γ so that it encircles the logarithmic branch cut from P_0 to Q_0. We actually decompose $\gamma = \gamma_1 + \alpha_{P_0} + \gamma_2 + \alpha_{Q_0}$ as indicated in figure 3. The discontinuity across the logarithmic cut is given by the residues f_0 and g_0 of $f \mathrm{d}z$ and $g \mathrm{d}z$ at P_0 in such a way that

$$
\int_{\gamma_1 + \gamma_2} pFG \, \mathrm{d}z = \int_{\beta_r} p \big(FG - (F - 2i\pi f_0)(G - 2i\pi g_0) \big) \mathrm{d}z .
\tag{A.8}
$$

Using the fact that the residues at infinity are given by minus the integral of the corresponding forms over $\alpha = \sum_i \alpha_i$, we thus obtain

$$
\mathscr{I} = - \oint_{\alpha} g \, \mathrm{d}z \int_{\beta_r} pF \, \mathrm{d}z - \oint_{\alpha} f \, \mathrm{d}z \int_{\beta_r} pG \, \mathrm{d}z + \oint_{\alpha_{P_0} + \beta_{P_0}} pFG \, \mathrm{d}z .
\tag{A.9}
$$

Putting (A.7) and (A.9) together, we find the fundamental formula

$$\sum_{i=1}^{r}\left[\int_{\beta_i} pF\mathrm{d}z \oint_{\alpha_i} g\,\mathrm{d}z + \int_{\beta_i} pG\mathrm{d}z \oint_{\alpha_i} f\mathrm{d}z - \int_{\beta_i} g\,\mathrm{d}z \oint_{\alpha_i} pF\mathrm{d}z - \int_{\beta_i} f\mathrm{d}z \oint_{\alpha_i} pG\mathrm{d}z\right]$$
$$+ \int_{\beta_r} g\,\mathrm{d}z \oint_{\alpha} pF\mathrm{d}z + \int_{\beta_r} f\mathrm{d}z \oint_{\alpha} pG\mathrm{d}z = \oint_{\alpha_{P_0}+\beta_{P_0}} pFG\,\mathrm{d}z\,. \tag{A.10}$$

The right-hand side of (A.10) is often conveniently rewritten in terms of the analytic continuations $\hat{F}$ and $\hat{G}$ as

$$\oint_{\alpha_{P_0}+\beta_{P_0}} pFG\,\mathrm{d}z = -\oint_{\alpha} p(FG + \hat{F}\hat{G})\mathrm{d}z\,. \tag{A.11}$$

A relation more akin to the standard Riemann bilinear relations is found when F is single-valued on $\mathcal{C}$, i.e. when (A.2) is satisfied. Equation (A.10) then reduces to

$$\sum_{i=1}^{r}\left[\int_{\beta_i} pF\mathrm{d}z \oint_{\alpha_i} g\,\mathrm{d}z - \int_{\beta_i} g\,\mathrm{d}z \oint_{\alpha_i} pF\mathrm{d}z\right] + \int_{\beta_r} g\,\mathrm{d}z \oint_{\alpha} pF\mathrm{d}z = -\oint_{\alpha} p(FG + \hat{F}\hat{G})\mathrm{d}z\,. \tag{A.12}$$

This latter formula can be deduced straightforwardly from the Riemann bilinear relations found in textbooks.

B. The solution in the rank one case

In this appendix, we discuss the explicit solution of the extended theory in the special case corresponding to

$$W(z) = \frac{1}{2}mz^2\,, \quad V(z) = \lambda_{-1} + \lambda_0 z + \frac{1}{2}\lambda_1 z^2\,. \tag{B.1}$$

In the usual case, for which $\lambda_1 = \lambda_0 = 0$, the theory has N confining vacua, the N-fold degeneracy corresponding to chiral symmetry breaking. When λ_0 and λ_1 are turned on, we are going to find generalizations of these vacua as well as new purely quantum solutions that go to infinity in the classical limit.

The generating functions (1.3) and (1.4) are given by

$$R(z) = \frac{N}{2}\left[\lambda_0 + \lambda_1 z + \frac{2 + 2s\lambda_1/m - \lambda_0 z - \lambda_1 z^2}{\sqrt{z^2 - 4s/m}}\right]\,, \tag{B.2}$$

$$S(z) = \frac{Nm}{2}\left[z - \sqrt{z^2 - 4s/m}\right]\,. \tag{B.3}$$

The variable s, which coincides with the gluino condensate, is given by the equation

$$\int_{\beta_1} R\,\mathrm{d}z = NV(\mu_0) - 2N\ln\mu_0 + 2i\pi k = N\int_{2\sqrt{s/m}}^{\mu_0} \frac{\lambda_1 z^2 + \lambda_0 z - 2s\lambda_1/m - 2}{\sqrt{z^2 - 4s/m}}\,. \tag{B.4}$$

In terms of the instanton factor (1.15), this is equivalent to

$$q = \left(\frac{s}{m}\right)^N e^{-N\lambda_1 s/m}\,. \tag{B.5}$$

Let us study the solutions to (B.5) when λ_1 is small. First, there are the usual N solutions with small corrections,

$$s = m e^{2i\pi k/N} q^{1/N} \Big(1 + \lambda_1 e^{2i\pi k/N} q^{1/N} + \mathcal{O}\big(\lambda_1^2\big) \Big), \quad 0 \le k \le N-1 \,. \qquad (B.6)$$

More interestingly, there is also an infinite set of solutions that have very large values of s, of the form

$$s \simeq -\frac{m}{N\lambda_1} \Big(\ln\big(\lambda_1^N q\big) + 2i\pi k \Big), \quad k \in \mathbb{Z} \,. \qquad (B.7)$$

In terms of the Yang-Mills coupling constant g_{YM} and theta angle ϑ, this takes the suggestive form

$$s \simeq \frac{m}{N\lambda_1} \left(\frac{8\pi^2}{g_{\mathrm{YM}}^2} - i\vartheta - \ln \lambda_1^N \right) . \qquad (B.8)$$

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